TOPOLOGICAL STABLE RANK OF INCLUSIONS OF UNITAL C*-ALGEBRAS

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Abstract. Let $1 \in A \subset B$ be an inclusion of C*-algebras of C*-index-finite type with depth 2. We try to compute topological stable rank of B ($= \mathrm{tsr}(B)$) when A has topological stable rank one. We show that $\mathrm{tsr}(B) \leq 2$ when A is a tsr boundedly divisible algebra, in particular, A is a C*-minimal tensor product $UHF \otimes D$ with $\mathrm{tsr}(D) = 1$. When G is a finite group and α is an action of G on UHF, we know that a crossed product algebra $UHF \rtimes_\alpha G$ has topological stable rank less than or equal to two.

These results are affirmative datum to a generalization of a question by B. Blackadar in 1988.

Contents

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1. Introduction

The notion of topological stable rank for a C*-algebra A, denoted by tsr(A), was introduced by M. Rieffel, which generalizes the concept of dimension of a topological space ([34]). He presented the basic properties and stability theorem related to K-Theory for C*-algebras. In [34] he proved that $\mathrm{tsr}(\mathrm{A}\times_\alpha\mathbb{Z}) \leq \mathrm{tsr}(\mathrm{A})+1$, and asked if an irrational rotation algebra A_θ has topological stable rank two. I. Putnum ([33]) gave a complete answer to this question, that is, $\mathrm{tsr}(\mathrm{A}_\theta) = 1$. Moreover, using the notion of approximate divisibility and U. Haggerup's striking result ([19]), B. Blackadar, A. Kumjian, and M. Rørdam ([6]) proved that every nonrational noncommutative torus has topological stable rank one. Naturally, we pose a question of the how to compute topological stable rank of $A \times_\alpha G$ for any discrete group G.

On the contrary, one of long standing problems is whether a fixed point algebra of a UHF C*-algebra by an action of a finite group G is an AF C*-algebra. O. Bratteli ([7]) proved that any fixed point algebra of an UHF-algebra by a product type action of a finite abelian group is an AF C*-algebra. In 1988, B. Blackadar ([4]) constructed a symmetry on the CAR algebra whose fixed point algebra is not an AF C*-algebra. Note that A. Kumjian ([27]) constructed a symmetry on a simple AF C*-algebra whose fixed point algebra is not an AF C*-algebra. Later, D. Evans and A. Kishimoto proved that for any compact group $G \neq \{e\}$ and $p \geq 2$, there exists an action of G on M_{p^∞} whose fixed point algebra is not an AF C*-algebra. All t hese constructions embodied expressing the AF C*-algebra A as an inductive limit $A_1 \to A_2 \to \cdots \to A = \lim A_n$, where each C*-algebra A_n is not an AF C*-algebra. This is related to the classification theory of simple unital AH-algebras ([12],[13],[15]). Indeed, applying the classification theory G. Elliott constructed a symmetry α on an UHF algebra, and proved that $UHF \rtimes_\alpha (\mathbb{Z}/2\mathbb{Z})$ is not AF C*-algebra, but AI-algebra, that is, the inductive limit of direct sums of $C([0,1]) \otimes M_n(\mathbb{C})$. Note that this crossed product algebra has topological stable rank one and real rank one. B. Blackadar proposed the question in the same article whether $\mathrm{tsr}(\mathrm{A} \times_\alpha \mathrm{G}) = 1$ for any unital AF C*-algebra A, a finite group G, and an action α of G on A.

In this paper we try to solve B. Blackadar's question from more general situation using C*-index theory defined by Y. Watatani ([36]). In the case that an inclusion $1 \in A \subset B$ is of index-finite type with depth 2 if A is tsr boundedly divisible algebra (see [35, Definition 4.1] and section 5) with $\mathrm{tsr}(\mathrm{A}) = 1$ we show that $\mathrm{tsr}(\mathrm{B}) \leq 2$ (Theorem 5.1). Hence if A is a UHF C*-algebra, we conclude that $\mathrm{tsr}(\mathrm{B}) \leq 2$ under the above condition. Therefore we get an affirmative data to B. Blackadar's question. Namely, for any UHF C*-algebra A, a finite group G, and an action α of G on A, we conclude that a crossed product algebra $A\times_\alpha G$ has $\mathrm{tsr}(\mathrm{A}\times_\alpha\mathrm{G}) \leq 2$. We can not still get the complete answer, but it seems to guarantee that the question would be solved affirmatively.

This paper is organized as follows. In section 2 we state a number of preliminary results about topological stable rank. In section 3 we explain a brief survey of C*-index theory. In section 4 we study the quasi-basis for the induced conditional expectation of the derived inclusion $1 \in pAp \subset pBp$ from the inclusion $1 \in A \subset B$ and a non-zero projection $p \in A$. We give a new estimate of topological stable rank for the inclusion of index-finite type with depth 2 in section 5 and give the main theorem, that is, that topological stable rank of a crossed product algebra of a UHF algebra by any finite group and any action has less than or equal to 2.

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2. Topological stable rank

In this section we present a definition of topological stable rank and some basic estimate formulas for it.

Definition 2.1. *Let A be a unital C*-algebra and $Lg_n(A)$ be the set of elements (b_i) of A^n such that*

$$Ab_1 + Ab_2 + \cdots + Ab_n = A.$$

Then topological stable rank of A, $\mathrm{tsr(A)}$, to be the least integer n such that the set $Lg_n(A)$ is dense in A^n. Topological stable rank of a non-unital C-algebra is defined by topological stable rank of its unitaization algebra $\tilde{A}$*

Note that $\mathrm{tsr(A)} = 1$ is equivalent to have the dense set of invertible elements in $\tilde{A}$. Here are some formulas for computing stable rank of C*-algebras.

Lemma 2.2. *Let A be a unital C*-algebra, and let $a_1, a_2, \ldots, a_n \in A$. The followings are equivalent:*

(1) *There are $c_1, c_2, \ldots, c_n \in A$ such that $c_1a_1 + c_2a_2 + \cdots + c_na_n$ is invertible.*
(2) *There are $c_1, c_2, \ldots, c_n \in A$ such that $c_1a_1 + c_2a_2 + \cdots + c_na_n = 1$.*
(3) *$a_1^*a_1 + a_2^*a_2 + \cdots + a_n^*a_n$ is invertible.*

Proof. Standard. □

Theorem 2.3 ([34]). (1) *Let J be a closed two-sided ideal of C*-algebra A. Then*

$$\mathrm{tsr(A)} \leq \max\{\mathrm{tsr(J)}, \mathrm{tsr(A/J)} + 1\}.$$

(2) *Let n be a positive integer. Then*

$$\mathrm{tsr(M_n(A))} = \left\{\frac{\mathrm{tsr(A)} - 1}{\mathrm{n}}\right\} + 1,$$

where $\{t\}$ denotes the least integer which is greater than or equal to t.

(3) *Let p be a non-zero projection in A. Then $\mathrm{tsr(A)} = 1$ if and only if $\mathrm{tsr(pAp)} = \mathrm{tsr((1-p)A(1-p))} = 1$.*
(4) *Let α be an action of A. Then*

$$\mathrm{tsr(A \rtimes_\alpha \mathbb{Z})} \leq \mathrm{tsr(A)} + 1.$$

(5) *Let $\mathbb{K}$ be a C*-algebra of compact operators on an infinite dimensional separable Hilbert space. Then*

$$\mathrm{tsr(A) = 1} \text{ if and only if } \mathrm{tsr(A \otimes \mathbb{K}) = 1}.$$

From the point of C*-module we have the following formula for topological stable rank.

Theorem 2.4. *Let $1 \in A \subset B$ be an inclusion of C*-algebras. Suppose that there are elements $\{v_i\}_{i=1}^n \in B$ such that*

$$B = Av_1 + Av_2 + \cdots + Av_n.$$

Then

$$\mathrm{tsr(B) \leq n \times tsr(A)}.$$

Proof. The proof is the same as in [23, Theorem 2.1]. We will put a sketch of its proof for a self-contained.

We first assume that $\mathrm{tsr(A) = 1}$. Let $\{a_{i1}v_1 + \cdots + a_{in}v_n \mid i = 1, \ldots, n\}$ be n elements in B. Then,

$$\begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$
$$= av$$

Since $\mathrm{tsr(M_n(A)) = 1}$ by Theorem 2.3 (2), we can approximate the matrix (a_{ij}) close enough by an invertible matrix $x = (x_{ij})$ in $M_n(A)$. Then the element $y = xv$ is close to av. Since $(v_1, v_2, \ldots, v_n) \in Lg_n(B)$, $(x_{11}v_1 + \cdots + x_{1n}v_n, \ldots, x_{n1}v_1 + \cdots + x_{nn}v_n)$ belongs to $Lg_n(B)$ by Lemma 2.2, and is close to $(a_{11}v_1 + \cdots + a_{1n}v_n, \ldots, a_{n1}v_1 + \cdots + a_{nn}v_n)$, which completes the proof in the case that $\mathrm{tsr(A) =}$ 1.

Similarly, one can prove the theorem when $\mathrm{tsr(A) > 1}$ using Theorem 2.3 (2). □

Corollary 2.5. *Let $1 \in A \subset B$ be an inclusion of C*-algebras, and let E be a faithful conditional expectation from B to A of index finite type. That is, there is a quasi-basis $\{(v_i, v_i^*)\}_{i=1}^n$ in $B \times B$ such that any element $x \in B$ can be written as*

$$x = \sum_{i=1}^n E(xv_i)v_i^* = \sum_{i=1}^n v_i E(v_i^* x).$$

Then

$$\mathrm{tsr(B) \leq n \times tsr(A)}.$$

Proof. Since

$$B = Av_1^* + Av_2^* + \cdots + Av_n^*,$$

we are done by Theorem 2.4. □

Corollary 2.6. *Let A be a unital C*-algebra and let G be a finite group. Then*

$$\mathrm{tsr(A \rtimes_\alpha G) \leq |G| tsr(A)},$$

where $|G|$ is the cardinal number of G.

Proof. Let $\alpha\colon G \to Aut(A)$ be a representation and we assume that the crossed product $A \rtimes_\alpha G$ acts on some Hilbert space. Let $\{u_g\}_{g\in G}$ be implemented unitaries of α_g such that $\alpha_g(a) = u_g a u_g^*$, for all $a \in A$. Then any element x in $A \rtimes_\alpha G$ can be written as $x = \sum_{g\in G} a_g u_g$. Let $E\colon A \rtimes_\alpha G \to A$ be the canonical conditional expectation by $E(x) = a_e$. Then,

$$x = \sum_{g\in G} E(xu_g^*)u_g, \quad \forall x \in A \rtimes_\alpha G.$$

Therefore it follows from Corollary 2.5 that

$$\mathrm{tsr}(\mathrm{A} \rtimes_\alpha \mathrm{G}) \le |\mathrm{G}|\mathrm{tsr}(\mathrm{A}).$$

□

Remark 2.7. *The estimate of topological stable rank of crossed product in* Corollary 2.6 *is not best one. Indeed, in the case of $G = \mathbb{Z}/n\mathbb{Z}$ we have*

$$\mathrm{tsr}(\mathrm{A} \rtimes_\alpha \mathrm{G}) \le \mathrm{tsr}(\mathrm{A}) + 1.$$

using Theorem 2.3 (4). *But, in section 5 we show more better estimate in the case that A is tsr boundedly divisible with* $\mathrm{tsr}(\mathrm{A}) = 1$, *$G$ any finite group, and α an action of G on A as follows:*

$$\mathrm{tsr}(\mathrm{A} \rtimes_\alpha \mathrm{G}) \le 2.$$

3. C*-Index Theory

In this section we summarize the C*-index theory of Y. Watatani ([36]).

Let $1 \in A \subset B$ be an inclusion of C*-algebras, and let $E\colon B \to A$ be a faithful conditional expectation from B to A.

A finite family $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ in $B \times B$ is called *a quasi-basis* for E if

$$\sum_{i=1}^n u_i E(v_i b) = \sum_{i=1}^n E(bu_i)v_i = b \text{ for } b \in B.$$

We say that a conditional expectation E is of *index-finite type* if there exists a quasi-basis for E. In this case the index of E is defined by

$$\mathrm{Index}(E) = \sum_{i=1}^n u_i v_i.$$

(We say also that the inclusion $1 \in A \subset B$ is of *index-finite type.*)

Note that Index(E) does not depend on the choice of a quasi-basis ([22, Example 3.14]) and every conditional expectation E of index-finite type on a C*-algebra has a quasi-basis of the form $\{(u_1, u_1^*), \ldots, (u_n, u_n^*)\}$ ([36, Lemma 2.1.6]). Moreover Index(E) is always contained in the center of B, so that it is a scalar whenever B has the trivial center, in particular when B is simple ([36, Proposition 2.3.4]).

Let $E\colon B \to A$ be a faithful conditional expectation. Then $B_A(= B)$ is a pre-Hilbert module over A with an A-valued inner product

$$\langle x, y\rangle = E(x^*y), \quad x, y \in B_A.$$

Let $\mathcal{E}$ be the completion of B_A with respect to the norm on B_A defined by

$$\|x\|_{B_A} = \|E(x^*x)\|_A^{1/2}, \quad x \in B_A.$$

Then $\mathcal{E}$ is a Hilbert C^*-module over A. Since E is faithful, the canonical map $B \to \mathcal{E}$ is injective. Let $L_A(\mathcal{E})$ be the set of all (right) A-module homomorphisms $T\colon \mathcal{E} \to \mathcal{E}$ with an adjoint A-module homomorphism $T^*\colon \mathcal{E} \to \mathcal{E}$ such that

$$\langle T\xi, \zeta\rangle = \langle \xi, T^*\zeta\rangle \quad \xi, \zeta \in \mathcal{E}.$$

Then $L_A(\mathcal{E})$ is a C^*-algebra with the operator norm $\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}$. There is an injective $*$-homomorphism $\lambda\colon B \to L_A(\mathcal{E})$ defined by

$$\lambda(b)x = bx$$

for $x \in B_A$ and $b \in B$, so that B can be viewed as a C^*-subalgebra of $L_A(\mathcal{E})$. Note that the map $e_A\colon B_A \to B_A$ defined by

$$e_A x = E(x), \quad x \in B_A$$

is bounded and thus it can be extended to a bounded linear operator, denoted by e_A again, on $\mathcal{E}$. Then $e_A \in L_A(\mathcal{E})$ and $e_A = e_A^2 = e_A^*$; that is, e_A is a projection in $L_A(\mathcal{E})$. A projection e_A is called the *Jones projection* of E.

The *(reduced) C^*-basic construction* is a C^*-subalgebra of $L_A(\mathcal{E})$ defined to be

$$C^*(B, e_A) = \overline{span\{\lambda(x)e_A\lambda(y) \in L_A(\mathcal{E}) : x,\ y \in B\ \}}^{\|\cdot\|}$$

([36, Definition 2.1.2]).

Then we have

Lemma 3.1. ([36, Lemma 2.1.4])

(1) $e_A C^*(B, e_A) e_A = \lambda(A) e_A$.
(2) $\psi : A \to e_A C^*(B, e_A) e_A$, $\psi(a) = \lambda(a) e_A$, *is a $*$-isomorphism (onto).*

Lemma 3.2. ([36, Lemma 2.1.5]) *The following are equivalent:*

(1) $E : B \to A$ *is of index-finite type.*
(2) $C^*(B, e_A)$ *has an identity and there exists a number c with $0 < c < 1$ such that*

$$E(x^*x) \geq c(x^*x) \quad x \in B.$$

The above inequality was shown first in [31] by Pimsner and Popa for the conditional expectation $E_N\colon M \to N$ from a type II_1 factor M onto its subfactor N (c can be taken as the inverse of the Jones index $[M : N]$).

A conditional expectation $E_B\colon C^*(B, e_A) \to B$ defined by

$$E_B(\lambda(x)e_A\lambda(y)) = (\mathrm{Index}(E))^{-1}xy \text{ for } x \text{ and } y \in B$$

is called *the dual conditional expectation* of $E : B \to A$. If E is of index-finite type, so is E_B with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = u_i e_A \mathrm{Index(E)}^{\frac{1}{2}}$, and $\{(u_i, u_i^*)\}$ is a quasi-basis for E ([36, Proposition 2.3.4]).

Even if $\mathrm{Index}(E)$ is scalar, we do not know the relation between the number of pairs in a quasi-basis and $\mathrm{Index}(E)$ ([22, Example 3.14][28, Lemma 3.4]). Izumi , however, showed recently that if we extend a conditional expectation E from σ-unital C*-algebra D onto stable simple C*-algebra C with $\overline{DC} = D$ to a faithful conditional expectation $\tilde{E}$ from the multiplier algebra $M(D)$ onto $M(C)$, then it has only one pair as a quasi-basis([22, Proposition 3.6]).

The inclusion $1 \in A \subset B$ of unital C*-algebras of index-finite type is said to have *finite depth* k if the derived tower obtained by iterating the basic construction

$$A' \cap A \subset A' \cap B \subset A' \cap B_2 \subset A' \cap B_3 \subset \cdots$$

satisfies $(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$, where $\{e_k\}_{k\geq 1}$ are projections derived obtained by iterating the basic construction such that $B_{k+1} = C^*(B_k, e_k)$ $(k \geq 1)$ $(B_1 = B, e_1 = e_A)$. Let $E_k : B_{k+1} \to B_k$ be a faithful conditional expectation correspondent to e_k for $k \geq 1$. Moreover we have

Lemma 3.3. ([36, Lemma 2.3.5]) *Suppose* $\mathrm{Index(E)}$ *is in* A. *Then*

$$\begin{cases} e_{k+1}e_ke_{k+1} = (\mathrm{Index(E)})^{-1}\mathrm{e_{k+1}} \\ e_ke_{k+1}e_k = (\mathrm{Index(E)})^{-1}\mathrm{e_k} \end{cases}$$

for $1 \leq k$.

When G is a finite group and α an action of G on A, it is well known that an inclusion $1 \in A \subset A \rtimes_\alpha G$ is of depth 2. We will give its proof for a self-contained.

Lemma 3.4. *Let A be a unital C*-algebra, G a finite group, and α an action of G on A. Then an inclusion $1 \in A \subset A \rtimes_\alpha G$ is of depth 2.*

Proof. Assume that $A \rtimes_\alpha G$ acts on some Hilbert space H, and there are implemented unitaries $\{u_g\}_{g\in G}$ such that $\alpha_g(a) = u_g a u_g^*$ and any element $x \in A \rtimes_\alpha G$ can be written by $x = \sum_{g\in G} a_g u_g$ for some $a_g \in A$. Let $E\colon A \rtimes_\alpha G \to A$ be the canonical conditional expectation by $E(\sum_{g\in G} a_g u_g) = a_e$. Then a set $\{(u_g^*, u_g)\}_{g\in G}$ is a quasi-basis for E. Note that $\mathrm{Index(E)} = |\mathrm{G}|$, where $|G|$ is the cardinal number of G.

Let

$$1 \in A \subset A \rtimes_\alpha G \subset B_2 \subset B_3 \subset \cdots$$

be a sequence of basic constructions.

Claim 1: $u_h e_A u_h^*$ is a projection from $\mathcal{E}_{A\rtimes_\alpha G}$ to Au_h for each $h \in G$.

Proof. It is obvious that $u_h e_A u_h^*$ is projection. Since

$$\begin{aligned} u_h e_A u_h^*(\textstyle\sum_g a_g u_g) &= u_h e_A \textstyle\sum u_h^* a_g u_g \\ &= u_h E(\alpha_{h^{-1}}(a_g) u_h^* u_g) \\ &= u_h \alpha_{h^{-1}}(a_h) = a_h u_h, \end{aligned}$$

we have the claim 1. □

Claim 2: For any $h \in G$ $u_h e_A u_h^* \in A' \cap B_2$.

Proof. Since for any $a \in A$

$$\begin{aligned} u_h e_A u_h^* a &= u_h e_A u_h^* a u_h u_h^* \\ &= u_h e_A \alpha_{h^{-1}}(a) u_h^* \\ &= u_h \alpha_{h^{-1}}(a) e_A u_h^* \\ &= a u_h e_A u_h^*, \end{aligned}$$

we have the claim 2. □

Claim 3: $\{u_h e_A u_h^*\}_{h\in G}$ are orthogonal projections in B_2.

Proof. Trivial.

Claim 4: $(A' \cap B_2)e_2(A' \cap B_2) = A' \cap B_3$, where e_2 is a projection correspondent to the dual conditional expectation from B_2 to B_1.

Proof. Note that $(A' \cap B_2)e_2(A' \cap B_2)$ is a closed two-sided ideal of $A' \cap B_3$ ([18, Theorem 4.6.3(ii)]). So we have only to show that this ideal contains an identity.

Since $u_g e_A u_g^* \in A' \cap B_2$, we have

$$\begin{aligned}(u_g e_A u_g^*)e_2(u_g e_A u_g^*) &= u_g e_A e_2 e_A u_g^* \\ &= \tfrac{1}{|G|} u_g e_A u_g^* \in (A' \cap B_2)e_2(A' \cap B_2).\end{aligned}$$

The last equality comes from Lemma 3.2. Hence

$$1 = \sum_{g\in G} u_g e_A u_g^* \in (A' \cap B_2)e_2(A' \cap B_2).$$

This means that the inclusion $A \subset A \rtimes_\alpha G$ is of depth 2. □

Remark 3.5. *When A is simple and α is outer, the crossed product $A \rtimes_\alpha G$ is simple* ([24]) *and we easily have*

(1) *$A' \cap B_2$ is isomorphic to $\sum_{g\in G} \mathbb{C}e_g$, where $e_g = u_g e_A u_g^*$*
(2) *$A' \cap B_3$ is isomorphic to $M_{|G|}(\mathbb{C})$.*

4. Quasi-basis for a conditional expectation

Let $1 \in A \subset B$ be an inclusion of unital C*-algebras and $E : B \to A$ be a faithful conditional expectation of index-finite type. Let $\{(v_i, v_i^*)\}_{i=1}^n$ be a quasi-basis for E.

The following is kindly informed by Y. Watatani ([37]) to the first author.

Proposition 4.1. *Under the above situation if a projection p in A has elements $\{y_j\}_{j=1}^m$ in A such that $\sum_{j=1}^m y_j p y_j^* = 1$, then a set $\{(pv_i y_j p, py_j^* v_i^* p)\}_{1\le i\le n, 1\le j\le m}$ is a quasi-basis for a conditional expectation $F_p = E|pBp$ from pBp onto pAp. Moreover* $\mathrm{Index(E)} = \mathrm{Index(F_p)}$.

Proof. It follows from the direct computation. Indeed for any $b \in B$ we have

$$\begin{aligned}\textstyle\sum_{i,j} pv_i y_j p F_p(py_j^* v_i^* ppbp) &= \textstyle\sum_{i,j} pv_i y_j p E(y_j^* v_i^* ppb)p \\ &= \textstyle\sum_{i=1}^n pv_i(\sum_{j=1}^m y_j p y_j^*)E(v_i^* ppb)p \\ &= \textstyle\sum_{i=1}^n pv_i E(v_i^* pb)p = p(pb)p = pbp.\end{aligned}$$

Similarly,

$$\sum_{i,j} F_p(pbppv_i y_j p) p y_j^* x_i^* p = pbp.$$

So $\{(pv_i y_j p, py_j^* x_i^* p)\}_{1\le i\le n, 1\le j\le m}$ is a quasi-basis for F_p and

$$\begin{aligned}\mathrm{Index}(F_p) = \textstyle\sum_{i,j} pv_i y_j p y_j^* x_i^* p &= \textstyle\sum_{i=1}^n pv_i(\sum_{j=1}^m y_j p y_j^*)v_i^* p \\ &= \textstyle\sum_{i=1}^n pv_i v_i^* p = (\mathrm{Index}(E))p.\end{aligned}$$

□

Corollary 4.2. *Let $1 \in A \subset B$ be an inclusion of unital C*-algebras and $E\colon B \to A$ be a faithful conditional expectation of index-finite type. Suppose that A is simple. Then for any non-zero projection p in A the conditional expectation $F_p = E|pBp\colon pBp \to pAp$ is of index-finite type.*

Proof. Since A is simple, there are finite elements a_i such that $\sum_i a_i p a_i^* = 1$. So the statement comes from the previous proposition. □

The following result shows that the Jones projection of F_p in the previous result is $e_A p$.

Proposition 4.3. *Let $1 \in A \subset B$ be an inclusion of C^*-algebras with finite index, and let e_A be the Jones projection correspondent to a faithful conditional expectation $E\colon B \to A$. Suppose that A is simple. Then for any projection $p \in A$, $e_A p$ is the Jones projection for the conditional expectation $F_p = E|pBp\colon pBp \to pAp$ and $pC^*(B, e_A)p$ is the basic construction for F_p.*

Proof. For any $x \in pBp$

$$e_A p x e_A p = E(px) e_A p = E(pxp) e_A p = F_p(x) e_A p.$$

Since A is simple, the map

$$pAp \ni x \to x e_A p (= x e_A) \in L(\mathcal{E})$$

is injective, where $\mathcal{E}$ is a Hilbert A-module obtained by the basic construction. Then by [36, Proposition 2.2.11] $e_A p$ is the Jones projection and $C^*(pBp, e_A p)$ is the basic construction for F. It is obvious that $C^*(pBp, e_A p) = pC^*(B, e_A)p$. □

The following result means that the number of elements in quasi-basis is stable under the particular situation.

Proposition 4.4. *Let $1 \in A \subset B$ be an inclusion of C^*-algebras with finite index. Suppose that there is a C^*-subalgebra D of A such that e_A is full in $D' \cap C^*(B, e_A)$. Then there are finitely elements $\{v_i\}_{i=1}^n$ in $D' \cap B$ such that for any non-zero projection $p \in D$ the sets $\{(v_i, v_i^*)\}_{i=1}^n$ and $\{(pv_i, pv_i^*)\}_{i=1}^n$ are quasi-basis for E and F_p, respectively.*

Proof.
Since $(D' \cap C^*(B, e_A)) e_A (D' \cap C^*(B, e_A)) = D' \cap C^*(B, e_A)$, there are finitely elements $\{x_i\}_{i=1}^n$ in $D' \cap C^*(B, e_A)$ such that

$$\sum_{i=1}^n x_i e_A x_i^* = 1.$$

Using the standard argument([31]) we can find $v_i \in B$ such that $v_i e_A = x_i e_A$. Since E is faithful, $v_i \in D' \cap B$.

Since for any $b \in B$

$$\begin{aligned} b = 1 \cdot b &= \textstyle\sum_{i=1}^n v_i e_A v_i^* b \\ &= \textstyle\sum_{i=1}^n v_i E(v_i^* b) = \sum_{i=1}^n E(b v_i) v_i^*, \end{aligned}$$

it follows that $\{(v_i, v_i^*)\}_{i=1}^n$ is a quasi-basis for E.

From the simple calculus we know that $\{(pv_i, pv_i^*)\}_{i=1}^n$ is a quasi-basis for F_p. □

5. Topological stable rank of inclusions of C*-algebras

In this section we prove the following main result:

Theorem 5.1. *Let $1 \in A \subset B$ be an inclusion of unital C*-algebras and $E\colon B \to A$ be a faithful conditional expectation of index-finite type. Suppose that the inclusion $1 \in A \subset B$ has depth 2 and A is tsr boundedly divisible with* $\mathrm{tsr(A)} = 1$*. Then B is tsr boundedly divisible. Moreover we have* $\mathrm{tsr(B)} \le 2$.

Recall that a C*-algebra A is *tsr boundedly divisible* ([35, Definition 4.1]) if there is a constant K (> 0) such that for every positive integer m there is an integer $n \ge m$ such that A can be expressed as $M_n(B)$ for a C*-algebra B with $\mathrm{tsr(B)} \le \mathrm{K}$. For any unital C*-algebra A with $\mathrm{tsr(A)} = 1$ a C*-minimal tensor product algebra $A \otimes UHF$ is a typical tsr boundedly divisible algebra.

Before giving the proof, we state a useful result by B. Blackadar.

Lemma 5.2. ([1, Lemma A6]) *Let A be a unital C*-algebra and p be a full projection in A such that $\sum_{i=1}^n u_i p v_i = 1$ for some elements u_i, v_i in A. Then* $\mathrm{tsr(A)} \le \mathrm{tsr(pAp)} + \mathrm{n} - 1$.

Remark 5.3. *Very recently, B. Blackadar sharpened the estimate in the previous lemma ([5]). That is, for any unital C*-algebra A and non-zero projection $p \in A$* $\mathrm{tsr(A)} \le \mathrm{tsr(pAp)}$.

The following estimate is the converse of Corollary 2.5.

Proposition 5.4. *Let $1 \in A \subset B$ be an inclusion of unital C*-algebra of index-finite type. Let $\{(v_i, v_i^*)\}_{i=1}^m$ be a quasi-basis for a faithful conditional expectation E from B onto A. Then we have*

$$\mathrm{tsr(A)} \le \mathrm{m}^2(\mathrm{tsr(B)} + 1) - 2\mathrm{m} + 1.$$

Proof. Let B_2 be the basic construction derived from this inclusion and E_2 be the dual conditional expectation from B_2 onto B. Note that a set $\{(v_i e_A (\mathrm{Index(E)})^{\frac{1}{2}}, (\mathrm{Index(E)})^{\frac{1}{2}} \mathrm{e_A v_i^*})\}_{\mathrm{i=1}}^{\mathrm{m}}$ is the quasi-basis for E_2. Hence from Corollary 2.5 we have

$$\mathrm{tsr(B_2)} \le \mathrm{m} \times \mathrm{tsr(B)}.$$

By [36, Lemma 3.3.4] there is an isomorphism $\varphi : B_2 \to qM_m(A)q$ such that

$$\phi(xe_A y) = [E(v_i^* x)E(yv_j)]_{i,j=1}^m,$$

where $q = [E(v_i^* v_j)]$. Note that q is a projection.

Claim: There are $X_i, Y_i \in M_m(A)$ such that $\sum_{i=1}^m X_i q Y_i = 1$.

Proof of the Claim: Let

$$(X_i)_{h,k} = \begin{cases} E(v_k) & \text{if } h = i \\ 0 & \text{others} \end{cases}, \quad (Y_i)_{h,k} = \begin{cases} E(v_h^*) & \text{if } k = i \\ 0 & \text{others} \end{cases}$$

for each $1 \le h, k \le m$.

Then from a simple calculation we have

$$\sum_{i=1}^m X_i q Y_i = 1.$$

So from Lemma 5.2 we have

$$\mathrm{tsr(M_m(A))} \le \mathrm{tsr(qM_m(A)q)} + \mathrm{m} - 1.$$

Since $\varphi(B_2)$ is isomorphic to $qM_m(A)q$ and $\mathrm{tsr(B_2) \leq m \times tsr(B)}$, we have

$$\mathrm{tsr(M_m(A)) \leq m \times tsr(B) + m - 1 = m(tsr(B)+1) - 1.}$$

Since from Theorem 2.3 (2) we have

$$\frac{\mathrm{tsr(A)}-1}{m} + 1 \leq \mathrm{tsr(M_m(A)) \leq m(tsr(B)+1) - 1,}$$

it follows that

$$\mathrm{tsr(A) \leq m^2(tsr(B)+1) - 2m + 1.}$$

□

Proof of Theorem 5.1
Let

$$1 \in A \subset B \subset B_2 \subset B_3 \subset \cdots$$

be the derived tower of iterating the basic construction and $\{e_k\}_{k\geq 1}$ be canonical projections such that $B_{k+1} = C^*(B_k, e_k)$, where $e_1 = e_A$. Since $1 \in A \subset B$ is of depth 2, we have

$$(A' \cap B_2)e_2(A' \cap B_2) = A' \cap B_3.$$

Hence there exit some $n \in \mathbb{N}$ a quasi-basis $\{(u_i, u_i^*)\}_{i=1}^n$ for the conditional expectation E_2 from B_2 onto B so that $u_i \in A' \cap B_2$ for $1 \leq i \leq n$ (see the proof of Proposition 4.4).

Since B_2 is stably isomorphic to A, we know that $\mathrm{tsr(B_2) = 1}$ by Theorem 2.3 (5). Take non-zero projection p in A. Since $u_i \in A' \cap B_2$ for $1 \leq i \leq n$, a set $\{(pu_i, u_i^*p)\}_{i=1}^n$ is a quasi-basis for the conditional expectation $F_p = E_2|pB_2p$ from pB_2p onto pBp. Hence from Proposition 5.4 we have

$$\begin{aligned}\mathrm{tsr(pBp)} &\leq n^2(\mathrm{tsr(pB_2p)}+1) - 2\mathrm{n} + 1\\ &= 2n^2 - 2n + 1.\end{aligned}$$

The last equality comes from that $\mathrm{tsr(B_2) = 1}$ and Theorem 2.3 (3).

Since A is tsr boundedly divisible, for any $l \in \mathbb{N}$ there are $k \in \mathbb{N}$ with $k \geq l$ and a C*-algebra D such that $A \cong M_k(D)$. Hence there is a matrix system $\{e_{i,j}\}_{i,j=1}^k$ in A such that $B \cong M_k(e_{1,1}Be_{1,1})$. Then from the above estimate we have

$$\mathrm{tsr(e_{1,1}Be_{1,1}) \leq n^2 + (n-1)^2.}$$

Therefore B is tsr boundedly divisible. From Theorem 2.3 (2) we can conclude that $\mathrm{tsr(B) \leq 2}$. □

Corollary 5.5. *Let A be a tsr boundedly divisible, unital C*-algebra with* $\mathrm{tsr(A)} = 1$, *G a finite group, and α an action of G on A. Then $A \rtimes_\alpha G$ is tsr boundedly divisible. Moreover we have* $\mathrm{tsr(A \rtimes_\alpha G) \leq 2}$.

Proof. Since the inclusion $1 \in A \subset A \rtimes_\alpha G$ is of index-finite type with depth 2 by Lemma 3.4, we can get the statement from Theorem 5.1. □

Corollary 5.6. *Let A be a unital C*-algebra with* $\mathrm{tsr(A)} = 1$, *G a finite group, and α an action of G on A. Then $(A \otimes UHF) \rtimes_{\alpha\otimes id} G$ is tsr boundedly divisible.*

Proof. Since $A \otimes UHF$ is tsr boundedly divisible and $\mathrm{tsr(A \otimes UHF)} = 1$ by using Theorem 2.3 (2), it comes from Corollary 5.5. □

When A in Corollary 5.6 is the trivial C*-algebra $\mathbb{C}$, we can get an affirmative data for B. Blackadar's question.

Corollary 5.7. *Let A be a UHF C*-algebra. Let G be a finite group, and α be an action of G on A. Then $A \rtimes_\alpha G$ is tsr boundedly divisible and*

$$\mathrm{tsr(A \rtimes_\alpha G)} \leq 2.$$

Remark 5.8. *The estimate in* Theorem 5.1 *is best possible. Indeed in* [4, Example 8.2.1] *B. Blackadar constructed an symmetry action α on CAR such that*

$$(C[0,1] \otimes CAR) \rtimes_{id \otimes \alpha} Z_2 \cong C[0,1] \otimes B,$$

where B is the Bunce-Deddens algebra of type 2^∞. Then since $K_1(B)$ is non-trivial, we know that

$$\mathrm{tsr(C[0,1] \otimes B)} = 2.$$

(*See also* [29, Proposition 5.2].)

Befor ending this section we present an inclusion $1 \in A \subset B$ with index 2 such that A is a tsr boundedly divisible and B can not be realized as some crossed product algebra of A by $\mathbb{Z}/2\mathbb{Z}$.

Let A be a unital C^*-algebra, α an action of $\mathbb{Z}/2\mathbb{Z}$ on A, and B be the crossed product $A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$. Let E the canonical conditional expectation and u an implemented unitary u such that $\alpha(u) = uau^*$ for $a \in A$ as in the proof of Lemma 3.4. Then $\{(1,1), (u^*, u)\}$ is a quasi-basis for E. Note that $E(u) = 0$. By [36, Lemma 3.3.4] there is a *-isomorphism $\varphi : C^*(B, e_A) \to qM_2(A)q$ such that

$$q = \begin{pmatrix} E(1 \cdot 1) & E(u^*) \\ E(u) & E(uu^*) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\varphi(xe_Ay) = \begin{pmatrix} E(x)E(y) & E(x)E(yu^*) \\ E(ux)E(y) & E(ux)E(yu^*) \end{pmatrix}$$

for $x, y \in B$. Here e_A is the Jones projection for the inclusion $A \subset B$. Therefore we can identify the basic construction with $M_2(A)$.

By this identification,

$$A \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & \alpha(a) \end{pmatrix} \mid a \in A \right\} \quad B \cong \left\{ \begin{pmatrix} a & b \\ \alpha(b) & \alpha(a) \end{pmatrix} \mid a, b \in A \right\}$$

and

$$\varphi(e_A) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi(1 - e_A) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $[\varphi(e_A)] = [1 - \varphi(e_A)]$ in $K_0(A)$.

From this observation we have

Lemma 5.9. *Let A be a unital C^*-algebra and α an action of $\mathbb{Z}/2\mathbb{Z}$ on A. Let B be the crossed product $A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ and e_A the Jones projection of the inclusion $A \subset B$. If $\varphi : C^*(B, e_A) \to M_n(A)$ is the canonical isomorphism, then we have $[\varphi(e_A)] = [1 - \varphi(e_A)]$ in $K_0(A)$.*

Proposition 5.10. *Let A be a simple unital C*-algebra. Suppose that p is a projection in A with $[p] \neq [\alpha(p)]$ in $K_0(A)$. Then the inclusion $pAp \subset pBp$ can not be represented as $pAp \subset pAp \times_\beta \mathbb{Z}/2\mathbb{Z}$ for any $\beta \in Aut(pAp)$.*

Proof. By the identification, $\varphi(p) = \begin{pmatrix} p & 0 \\ 0 & \alpha(p) \end{pmatrix}$. The Jones projection for the conditional expectation $F_p = E|_{pBp} : pBp \to pAp$ is $e_A p$ by Proposition 4.3, so $\varphi(e_A p) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. Then $\varphi(p - e_A p) = \begin{pmatrix} 0 & 0 \\ 0 & \alpha(p) \end{pmatrix}$. By the assumption, $[\varphi(e_A p)] \neq [\varphi(p - e_A p)]$ in $K_0(A)$ and hence $[\varphi(e_A p)] \neq [\varphi(p - e_A p)]$ in $K_0(pAp)$. So we have the conclusion by Lemma 5.9. □

Remark 5.11. *When $A = A_1 \oplus A_2$ for simple unital C*-algebras A_1 and A_2, we can also get the same conclusion in Proposition 5.10. Indeed, $e_A p$ becomes the Jones projection of the inclusion $pAp \subset pBp$ as the same argument in Proposition 4.3.* □

The following example is due to T. Katsura and N. C. Phillips.

Example 5.12. *Let α be an automorphism on $CAR \otimes \mathbb{K}$ such that $[\alpha(1 \otimes e_0)] \neq [1 \otimes e_0]$ in $K_0(CAR)$, where e_0 is a minimal projection. Such an automorphism can be constructed by modifying the shift operator in [25]. Set D as the unitaization of $CAR \otimes \mathbb{K}$. Then α can be an automorphism on D. We call it α again.*

Define a symmetry γ on $D \oplus D$ by $\gamma((a, b)) = (\alpha^{-1}(b), \alpha(a))$ for (a, b) in $D \oplus D$. Consider the inclusion

$$D \oplus D \subset (D \oplus D) \rtimes_\gamma \mathbb{Z}/2\mathbb{Z}.$$

Since

$$\begin{aligned} [\gamma((1 \otimes e_0, 1 \otimes e_0))] &= [(\alpha^{-1}(1 \otimes e_0), \alpha(1 \otimes e_0)] \\ &\neq [(1 \otimes e_0, 1 \otimes e_0)] \end{aligned}$$

in $K_0(D \oplus D)$ by the construction. We know that the inclusion

$$(1 \otimes e_0, 1 \otimes e_0)(D \oplus D)(1 \otimes e_0, 1 \otimes e_0) \subset (1 \otimes e_0, 1 \otimes e_0)((D \oplus D) \rtimes_\gamma \mathbb{Z}/2\mathbb{Z})(1 \otimes e_0, 1 \otimes e_0)$$

can not be represented as

$$(1 \otimes e_0, 1 \otimes e_0)(D \oplus D)(1 \otimes e_0, 1 \otimes e_0) \subset (1 \otimes e_0, 1 \otimes e_0)(D \oplus D)(1 \otimes e_0, 1 \otimes e_0) \rtimes_\beta \mathbb{Z}/2\mathbb{Z}$$

for any $\beta \in Aut((1 \otimes e_0, 1 \otimes e_0)(D \oplus D)(1 \otimes e_0, 1 \otimes e_0))$ by Proposition 5.10 and Remark 5.11. Note that

$$(1 \otimes e_0, 1 \otimes e_0)(D \oplus D)(1 \otimes e_0, 1 \otimes e_0) \cong CAR \oplus CAR,$$

that is, the algebra is tsr boundedly divisible. □

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