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TRACIALLY QUASIDIAGONAL EXTENSIONS AND TOPOLOGICAL STABLE RANK

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Abstract. It is known that a unital simple C^*-algebra A with tracial topological rank zero has real rank zero and topological stable rank one. In this note we construct unital C^*-algebras with tracial topological rank zero but topological stable rank two.

1. Introduction

The tracial topological rank of a unital C^*-algebra A, denoted by $\mathrm{TR}(A)$, was introduced as a noncommutative analog of the covering dimension for a topological space X ([Ln2] and [Ln3]; see also Definition 4.2 below). It plays an important role in the classification of amenable C^*-algebras (see [Ln3], [Ln5] and [Ln6]). As in the case of real rank (see [BP]), a unital commutative C^*-algebra $C(X)$ has tracial topological rank k if and only if $\dim X = k$. It was shown in [HLX1] that if $\dim X = k$ and $\mathrm{TR}(A) = m$ then $\mathrm{TR}(C(X)\otimes A) \le k+m$. (For the case of real rank see [NOP, Corollary 1.10].) At present, the most interesting cases are C^*-algebras with tracial topological rank at most one. If A is a unital separable simple C^*-algebra with tracial topological rank zero, it was shown in [Ln4] that A is quasidiagonal, has real rank zero, topological stable rank one and weakly unperforated $K_0(A)$.

We are also interested in C^*-algebras that are not simple. Let

$$(*) \qquad 0 \to J \to E \to A \to 0$$

be a short exact sequence with $\mathrm{TR}(A) = 0$ and $\mathrm{TR}(eJe) = 0$ for every projection $e \in A$. In [HLX2] it was shown that $\mathrm{TR}(E) = 0$ if and only if the extension is tracially quasidiagonal. (For the definition of a tracially quasidiagonal extension see Definition 5.1.) Quasidiagonal extensions are tracially quasidiagonal. A natural question is whether there exist any tracially quasidiagonal extensions which are not quasidiagonal. Very recently, the first author showed that such extensions do indeed exist. The example given also shows that there are C^*-algebras with $\mathrm{TR}(E) = 0$ such that real rank of E is not

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zero. The specific extension $(*)$ given there has the property that both J and A (are simple and) have topological stable rank one. Moreover, the index map $\partial_0 : K_0(A) \to K_1(J)$ is not trivial. It is known that, in this case, E has topological stable rank one if the index map $\partial_1 : K_1(A) \to K_0(J)$ is zero (see [Ni, Lemma 3] and [LR, Proposition 4]). A natural question is whether there is a tracially quasidiagonal extension with $\partial_1 \neq 0$. In this note we construct a tracially quasidiagonal extension $0 \to J \to E \to A \to 0$ of C^*-algebras such that J is a simple AF-algebra, A is a simple AT-algebra with real rank zero and $\partial_1 \neq 0$. That is, this extension is not a quasidiagonal extension. Moreover, E is a quasidiagonal C^*-algebra which has tracial topological rank zero and real rank zero, but topological stable rank two. To construct such an example we use the K_0-embedding property (see [BD]) and the generalized inductive limits in the sense of [BE].

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2. Preliminaries

Definition 2.1. Let A be a separable C^*-algebra. Then A is called quasidiagonal (QD) if there exists a faithful representation $\pi : A \to B(H)$ and an increasing sequence of finite rank of projections, $P_1 \leq P_2 \leq P_3 \leq \cdots$, such that $\|P_n\pi(a) - \pi(a)P_n\| \to 0$ for all $a \in A$ and $P_n \to 1_H$ in the strong operator topology as $n \to \infty$.

If A is a QD C^*-algebra, then there is an injective homomorphism which maps A into $\prod_n M_{k(n)}/\oplus M_{k(n)}$ for some increasing sequence $\{k(n)\}$.

Recently N. Brown and M. Dadarlat studied extensions of quasidiagonal C^*-algebras. Suppose that

$$0 \longrightarrow J \longrightarrow E \longrightarrow A \longrightarrow 0$$

is a short exact sequence of separable C^*-algebras such that A and J are QD C^*-algebras, and A is nuclear and satisfies the Universal Coefficient Theorem. Brown and Dadalart [BD] showed that if the indexes $\partial_i : K_i(A) \to K_i(J)$ are both zero then E is a QD C^*-algebra. Moreover, they proved that for QD C^*-algebras J satisfying a certain property (namely, the K_0-embedding) the C^*-algebra E determined by the extension

$$0 \longrightarrow J \longrightarrow E \longrightarrow B \longrightarrow 0$$

with a QD C^*-algebra B is again QD if and only if E is stably finite. It follows from a result of J. Spielberg (see [Sp, Lemma 1.14]) that this is the case when $\partial_1(K_1(B)) \cap K_0(J)_+ = \{0\}$. Spielberg [Sp, Lemma 1.14] also showed that all AF-algebras satisfy the (K_0-embedding) property.

Let J be a stable (non-elementary) simple AF-algebra. Let $\rho : K_0(J) \to \mathrm{Aff}(T(J))$, where $T(J)$ is the compact tracial space $\{\tau : \tau(p) = 1\}$ for some nonzero projection $p \in J$, be defined by $\rho([q]) = \tau(q)$ for $q \in K_0(J)$ and $\tau \in T(J)$. Since we have $K_1(M(J)/J) \cong K_0(J)$, there exists a unitary u in $M(J)/J$ (or in $M_k(M(J)/J)$) such that $[u] \in \ker \rho$. This unitary u together with J give the essential extension

$$0 \longrightarrow J \longrightarrow E \longrightarrow C(\mathbb{T}) \longrightarrow 0,$$

so that $\partial_1(K_1(C(\mathbb{T})))$ is nonzero but generated by $[u]$, $[u] \in \ker \rho$. The important fact that we are using is that $\partial_1(K_1(C(\mathbb{T}))) \cap K_0(J)_+ = \{0\}$. Therefore, by the above remarks, E is QD but the stable rank of A is not one (by [Ni, Lemma 3]).

3. Construction

Definition 3.1. We now define a very special simple AF-algebra. Let I_0 be a unital separable simple AF C^*-algebra with $K_0(I_0) = \mathbb{Q} \oplus \mathbb{Z}$ and $K_0(I_0)_+ = \{(r, m) : r \in \mathbb{Q}_+ \backslash \{0\}, m \in \mathbb{Z}\} \cup \{(0, 0)\}$. Let $B = C(\mathbb{T})$. Write $K_i(B) = \mathbb{Z}$ $(i = 0, 1)$. Take $u \in M(A)/A$ such that $[u] = (0, 1)$ in $\mathbb{Q} \oplus \mathbb{Z}$. Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. Set $I = I_0 \otimes \mathbb{K}$. As in Section 2, we obtain an essential extension

$$0 \longrightarrow I \longrightarrow E_1 \xrightarrow{\pi} B \longrightarrow 0,$$

such that $\partial_1([z]) = (0, 1)$ (in $\mathbb{Q} \oplus \mathbb{Z}$), where z is the canonical generator for B. As in Section 2, E_1 is a QD C^*-algebra.

Let $\{e_{ij}\}$ be the matrix unit for $\mathbb{K}$, and set $e_n = \sum_{i=1}^n e_{ii}$, $n = 1, 2, \ldots$ Here we identify e_{11} with 1_{I_0}.

Definition 3.2. Let A be a C^*-algebra, $\mathcal{G} \subset A$ a finite subset of A and $\varepsilon > 0$ a positive number. Recall that a positive linear map $L : A \to B$ (where B is a C^*-algebra) is said to be $\mathcal{G}$-ε-multiplicative if

$$\|L(a)L(b) - L(ab)\| < \varepsilon,$$

for all $a, b \in \mathcal{G}$.

The following result is certainly known (cf. [Ln7, Proposition 2.3]).

Proposition 3.3. *Let $A = M_n$. For any $\varepsilon > 0$ there is $\delta > 0$ such that if $L : A \to B$, where B is a unital C^*-algebra, is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, where $\mathcal{G}$ contains the matrix unit $\{e_{ij}\}_{i,j=1}^n$, then there is a homomorphism $h : A \to B$ such that*

$$\|L - h\| < \varepsilon.$$

Definition 3.4. We denote by δ_n the value of δ corresponding to $\varepsilon = 1/2^n$ in Proposition 3.3. We may assume that $0 < \delta_{n+1} < \delta_n < 1$.

DEFINITION 3.5. For any $k \in \mathbb{N}$ we let $\pi_k : M_k(E_1) \to M_k(B)$ denote the quotient map induced by π.

Let $\{\xi_1, \xi_2, \dots\}$ be a dense sequence of $\mathbb{T}$, where each point is repeated infinitely many times. Let $\{a_1, a_2, \dots\}$ be a dense sequence in the unit ball of E_1. Let $\mathcal{G}_n = \{0, a_1, \dots, a_n\}$, $n = 1, 2, \dots,$ and let $\mathcal{F}_1 = \mathcal{G}_1$. Write $I = \overline{\bigcup_{n=1}^{\infty} S_n}$, where the S_n are finite dimensional C^*-algebras and $S_n \subset S_{n+1}$. Let $\mathcal{S}_1$ be a finite subset of S_1 and let $\eta_1 > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital C^*-algebra B' there is a homomorphism $\bar{h}_1 : S_1 \to B'$ such that

$$\|L'|_{S_1} - \bar{h}_1\| < 1/2,$$

for any $\mathcal{S}_1$-η_1-multiplicative contractive completely positive linear map $L' : E \to B'$.

Let $\mathcal{F}_1 = \mathcal{G}_1 \cup \mathcal{S}_1$. Since E_1 is quasidiagonal, there is a (unital) contractive completely positive linear map $\psi_1 : E_1 \to M_{k(1)}$, which is $\mathcal{F}_1$-$1/2 \cdot 1/2 \cdot \eta_1/2^2$-multiplicative with $\|\psi_1(a)\| \geq (1/2)\|a\|$ for all $a \in \mathcal{F}_1$. Let $p_1 = 1_{I_0} \otimes e_{k(1)}$, where $e_{k(1)}$ is a rank $k(1)$ projection in $\mathbb{K}$. So $p_1 \in I$ and ψ_1 may be viewed as a map from E_1 to $p_1(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_1 \cong M_{k(1)}$. Put $\phi_1(a) = \pi(a)(\xi_1) \cdot (1_{E_1} - p_1)$ for $a \in E_1$. Let $C_1 = \phi_1(E) \oplus p_1(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_1$ and $C_1' = p_1(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_1$. Define $L_1 : E_1 \to E_2 = M_2(E_1)$ by

$$L_1(a) = \mathrm{diag}(a, \phi_1(a), \psi_1(a)),$$

for $a \in E_1$. Note that $L_1(S_1)$ is contained in $S_1 \oplus C_1'$. Moreover, $S_2 \otimes M_2 + C_1'$ is contained in a finite dimensional C^*-subalgebra of $M_2(I)$. Let C_1'' be the finite dimensional C^*-subalgebra generated by $S_2 \otimes M_2 + C_1'$. Since $S_1 \subset S_2$, we have $L_1(S_1) \subset C_1''$. Let $\mathcal{S}_2$ be a finite subset of C_1'' and let $1 > \eta_2 > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital C^*-algebra B' there exists a homomorphism $\bar{h}_2 : C_1'' \to B'$ such that

$$\|L'|_{C_1''} - \bar{h}_2\| < 1/2^2,$$

for any $\mathcal{S}_2$-η_2-multiplicative contractive completely positive linear map $L' : E_2 \to B'$.

Set $I_2 = M_2(I)$. Let $\mathcal{F}_2$ be a finite subset of E_2 containing $L_1(\mathcal{F}_1)$, $\{(a_{ij})_{i,j=1}^3 : a_{ij} = 0, a_1 \text{ or } a_2\}$, I_{C_2}, the standard generators of C_1, $\mathcal{S}_2$, and another matrix unit $\{u_{ij}\}_{i,j=1}^2$, where u_{11} and u_{22} are identified with $\mathrm{diag}(1_{E_1}, 0)$, $\mathrm{diag}(0, 1_{E_1})$. Since E_1 (unital) is quasidiagonal, there is a $\mathcal{F}_2$-$1/3 \cdot 1/2^2 \cdot \eta_2 \cdot \delta_{k(1)}/2^2$-multiplicative contractive completely positive linear map $\psi_2 : E_2 \to M_{k(2)}$ such that $(\psi_2)|_{M_2(\mathbb{C} \cdot 1_E)}$ is a homomorphism and $\|\psi_2(a)\| \geq (1 - 1/4)\|a\|$ for all $a \in \mathcal{F}_2$, and such that there is homomorphism $h_2 : C_1 \to M_{k(2)}$ such that

$$\|(\psi_2)|_{C_1} - h_2\| < 1/4.$$

Note that such a map h_2 exists by Proposition 3.3. We may assume that $k(2) > k(1) > 2$.

Let $E_3 = M_{2+1}(E_2) = M_{3!}(E_1)$ and $I_3 = M_{2+1}(I_2)$. Let $p_2' = 1_{I_0} \otimes e_{k(2)}$ and $p_2 = \mathrm{diag}(p_2', p_2') \in I_2$. Define $\phi_1^{(2)}(a) = \pi_2(a)(\xi_1) \cdot 1_{E_2}$ for $a \in E_2$, where the image of $\phi_1^{(2)}$ is identified with $M_2(\mathbb{C} \cdot 1_{E_2})$. Define $\phi_2(a) = \pi_2(a)(\xi_2) \cdot (1_{E_1} - p_2')$ for $a \in E_2$, where the image of ϕ_2 is identified with $M_2(\mathbb{C} \cdot (1_{E_1} - p_2'))$. Let $\Psi_2(a) = \mathrm{diag}(\psi_2(a), \psi_2(a))$ for $a \in E_2$. We now view Ψ_2 as a map $\Psi_2 : E_2 \to p_2(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_2 \subset p_2 I_2 p_2$. In particular, $\Psi_2(1_{E_2}) = p_2$. Define $L_2 : E_2 \to E_3$ by

$$L_2(a) = \mathrm{diag}(a, \phi_1^{(2)}(a), \phi_2(a), \Psi_2(a)).$$

It should be noted that the part $\mathrm{diag}(\phi_2(a), \Psi_2(a))$ is in E_2 and L_2 is unital. Let

$$C_2 = \phi_1^{(2)}(E_1) \oplus \phi_2(E_2) \oplus p_2 M_2(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_2,$$

and $C_2' = p_2(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_2$. Note that $L_2(C_1'')$ is contained in $C_1'' \oplus C_2'$ and $S_3 \otimes M_{3!} + C_2'$ is contained in a finite dimensional C^*-subalgebra of I_3. Let C_2'' be the finite dimensional C^*-subalgebra generated by $S_3 \otimes M_{3!} + C_2'$. Then, since $S_2 \subset S_3$ and $C_1' \subset C_2'$, we have $L_2(C_1'') \subset C_2''$.

Let $\mathcal{S}_3$ be a finite subset of C_2'' and let $\eta_3 > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital C^*-algebra B' there exists a homomorphism $\bar{h}_3 : C_2'' \to B'$ such that

$$\|L'|_{C_2''} - \bar{h}_3\| < 1/2^3,$$

for any $\mathcal{S}_3$-η_3-multiplicative contractive completely positive linear map $L' : E_3 \to B'$.

Let $E_4 = M_4(E_3)$ and $I_4 = M_4(I_3)$. Let $\mathcal{D}_2$ be a finite subset of C_2 containing 1_{C_2} and the standard generators of C_2. Let $\mathcal{F}_3$ be a finite subset of E_3 containing $L_2(\mathcal{F}_2)$, $\{(a_{ij})_{i,j=1}^{3\times 2} : a_{ij} = 0, a_1, a_2, \text{ or } a_3\}$, $\mathcal{D}_2$, $\mathcal{S}_3$ and another matrix unit $\{u_{ij}\}_{i,j=1}^{3}$, where u_{ii} is identified with a diagonal element with 1_{E_2} in the ith position and zero elsewhere.

Since E_1 is quasidiagonal and $E_3 = M_{3!}(E_1)$, there is a $\mathcal{F}_3$-$1/4 \cdot 1/2^3 \cdot \eta_3 \cdot \delta_{\dim C_2}/2^3$- multiplicative contractive completely positive linear map $\psi_3 : E_3 \to M_{k(3)}$ (where $k(3) > k(1) + k(2)$) such that $(\psi_3)|_{M_{3\cdot 2}(\mathbb{C}\cdot 1_E)}$ is a homomorphism, and there is a homomorphism $h_3 : C_2 \to M_{k(3)}$ such that

$$\|\psi_3|_{C_2} - h_3\| < 1/2^3.$$

Define $\phi_i^{(3)}(a) = \pi_{3!}(a)(\xi_i)$ for $a \in E_3$, where the image of $\phi_i^{(3)}$ is identified with $M_{3!}(\mathbb{C} \cdot 1_{E_1})$, $i = 1, 2$. Let $p_3' = 1_{I_0} \otimes e_{k(3)}$ and $p_3 = \mathrm{diag}(p_3', \ldots, p_3')$, where p_3' is repeated $3!$ times.

Thus, $p_3 \in I_3$. Let $\Psi_3(a) = \mathrm{diag}(\psi_3(a), \ldots, \psi_3(a))$ for $a \in E_3$, where $\psi_3(a)$ is repeated $3!$ times. We view Ψ_3 as a map $\Psi_3 : E_3 \to p_3 M_{3!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_3$.

Define $\phi_3(a) = \pi_{3!}(a)(\xi_3)$, where the image of ϕ_3 is identified with $M_{3!}(\mathbb{C} \cdot (1_{E_1} - p_3'))$. Define $L_3 : E_3 \to E_4$ by (for any $a \in E_3$)

$$L_3(a) = \mathrm{diag}(a, \phi_1^{(3)}(a), \phi_2^{(3)}(a), \phi_3(a), \Psi_3(a)).$$

Note that $\mathrm{diag}(\phi_3(a), \Psi_3(a)) \in E_3$. Put

$$C_3 = \oplus_{i=1}^2 \phi_i^{(3)}(E_3) \oplus \phi_3(E_3) \oplus p_3 M_{3!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_3,$$
$$C_3' = p_3 M_{3!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_3.$$

Let C_3'' be the finite dimensional C^*-algebra generated by $S_4 \otimes M_{4!} + C_3'$. Then, since $L_2(C_2'')$ is contained in $C_2'' \oplus C_3'$, $S_3 \subset S_4$ and $C_2' \subset C_3'$, we have $L_2(C_2'') \subset C_3''$.

We continue the construction in this fashion. With $C_n = \oplus_{i=1}^{n-1} \phi_i^{(n)}(E_n) \oplus \phi_n(E_n) \oplus p_n M_{n!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_n$, let $E_{n+1} = M_{n+1}(E_n)$, $I_{n+1} = M_{n+1}(I_n)$ and $C_n' = p_n M_{n!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_n$, and let C_n'' be the finite dimensional C^*–subalgebra generated by $S_{n+1} \otimes M_{(n+1)!} + C_n'$. Let $\mathcal{S}_{n+1}$ be a finite set in C_n'' and let $1 > \eta_{n+1} > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital C^*-algebra B' there exists a homomorphism $\bar{h}_{n+1} : C_n'' \to B'$ such that $\|L'|C_n'' - \bar{h}_{n+1}\| < 1/2^{n+1}$ for any $\mathcal{S}_{n+1}$-η_{n+1}-multiplicative contractive completely positive map $L' : E_{n+1} \to B'$.

Let $\mathcal{D}_n$ be a finite subset of C_n containing 1_{C_n} and the standard generators of C_n, and let $\mathcal{F}_{n+1}$ be a finite subset of E_{n+1} containing $L_n(\mathcal{F}_n)$, $\mathcal{S}_{n+1}$, $\{(a_{ij})_{i,j=1}^{(n+1)!} : a_{ij} = 0, a_1, \ldots, \mathrm{or} a_n\}$, $\mathcal{D}_n$ and a matrix unit $\{u_{ij}\}_{i,j=1}^{1+n}$, where u_{ii} is identified with $\mathrm{diag}(0, \ldots, 0, 1_{E_n}, 0, \ldots, 0)$ (with 1_{E_n} in the ith position). Since E_1 is quasidiagonal and $E_{n+1} = M_{(n+1)!}(E_1)$, there is a $\mathcal{F}_{n+1}$-$1/(n+2) \cdot 1/2^{n+1} \cdot \eta_{n+1} \cdot \delta_{\dim C_n}/2^{n+1}$-multiplicative contractive completely positive linear map $\psi_{n+1} : E_{n+1} \to M_{k(n+1)}$ such that $(\psi_n)|_{M_{(n+1)!}}(\mathbb{C} \cdot 1_{E_1})$ is a homomorphism, and there is a homomorphism, $h_{n+1} : C_n \to M_{k(n+1)}$ such that

$$\|(\psi_{n+1})|_{C_n} - h_{n+1}\| < 1/2^{n+1}.$$

Define $\phi_i^{(n+1)}(a) = \pi_{(n+1)!}(a)(\xi_i)$ for $a \in E_{n+1}$ and identify the image of $\phi_i^{(n+1)}$ with $M_{(n+1)!}(\mathbb{C} \cdot 1_{E_1})$, $i = 1, 2, \ldots, n$. Let $p_{n+1}' = 1_{I_0} \otimes e_{k(n+1)}$ and $p_{n+1} = \mathrm{diag}(p_{n+1}', \ldots, p_{n+1}')$, where p_{n+1} is repeated $(n+1)!$ times. Put $\Psi_{n+1}(a) = \mathrm{diag}(\psi_{n+1}(a), \ldots, \psi_{n+1}(a))$, where $\psi_{n+1}(a)$ is repeated $(n+1)!$ times. Thus the image of Ψ_{n+1} can be identified with $p_{n+1} M_{(n+1)!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_{n+1}$. Note that $\Psi_{n+1}(1_{E_{n+1}}) = p_{n+1}$. Define $\phi_{n+1}(a) = \pi_{(n+1)!}(a)(\xi_i)$, with its image identified with $M_{(n+1)!}(\mathbb{C} \cdot (1_{E_1} - p_{n+1}'))$. Note that the unit of $M_{(n+1)!}(\mathbb{C} \cdot (1_{E_1} - p_{n+1}'))$ is $1_{E_{n+1}} - p_{n+1}$. Define

$$L_{n+1}(a) = \mathrm{diag}(a, \phi_1^{(n+1)}(a), \phi_2^{(n+1)}(a), \ldots, \phi_n^{(n+1)}(a), \phi_{n+1}(a), \Psi_{n+1}(a)),$$

where $a \in E_{n+1}$. Let

$$C_{(n+1)} = \oplus_{i=1}^{n} \phi_i^{(n+1)}(E_{n+1}) \oplus \phi_{n+1}(E_{n+1}) \\ \oplus p_{n+1} M_{(n+1)!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_{n+1}$$

and

$$C'_{n+1} = p_{n+1} M_{(n+1)!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_{n+1}.$$

Let C''_{n+1} be the finite dimensional C^*-algebra generated by $S_{n+2} \otimes M_{(n+2)!} + C'_{n+1}$. Then, since $L_{n+1}(C''_n)$ is contained in $C''_n \oplus C'_{n+1}$, $S_{n+1} \subset S_{n+2}$, and $C'_n \subset C'_{n+1}$, we have $L_{n+1}(C''_n) \subset C''_{n+1}$. Let $\mathcal{S}_{n+2}$ be a finite set in C''_{n+1} and let $1 > \eta_{n+2} > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital C^*-algebra B' there exists a homomorphism $\bar{h}_{n+2} : C''_{n+1} \to B'$ such that $\|L'|C''_{n+1} - \bar{h}_{n+2}\| < 1/2^{n+2}$ for any $\mathcal{S}_{n+2}$-η_{n+2}-multiplicative contractive completely positive map $L' : E_{n+2} \to B'$.

It is easy to verify that (E_n, L_n) forms a general inductive limit in the sense of [BE]. Denote by E the C^*-algebra defined by this inductive limit. We will write $L_{n,n+k} : E_n \to E_{n+k}$ for the decomposition $L_{n+k-1} \circ \cdots \circ L_n$ and $L_{n,\infty} : E_n \to E$ for the map induced by the inductive limit. We will also use the fact that $\|L_n(a)\| = \|a\| = \|L_{n,\infty}(a)\|$ for all $a \in E_n$, $n = 1, 2, \ldots$.

Let $I_1 = I$, $I_{n+1} = M_{(1+n)!}(I)$. Then $I_n \cong I_0 \otimes \mathbb{K}$ and I_n is an ideal of E_n. Set $J_0 = \bigcup_{n=1}^{\infty} L_{n,\infty}(I_n)$ and $J = \bar{J}_0$.

Proposition 3.6. *J is an ideal of E.*

Proof. Let $a \in E$ and $b \in J$. We want to show that $ab, ba \in J$. For any $\varepsilon > 0$, there are $a' \in \bigcup_{n=1}^{\infty} L_{n,\infty}(E_n)$ and $b' \in J_0$ such that $\|a - a'\| < \varepsilon$ and $\|b - b'\| < \varepsilon$. It suffices to show that $a'b', b'a' \in J$. To simplify notation, without loss of generality, we may assume that $a \in \bigcup_{n=1}^{\infty} L_{n,\infty}(E_n)$ and $b \in J_0$. Therefore, there is an integer $n > 0$ such that $a = L_{n,\infty}(a_1)$ and $b = L_{n,\infty}(b_1)$, where $a_1 \in E_n$ and $b_1 \in I_n$. Moreover, there is an integer $N > n$ such that

$$\|L_{N,N+k} \circ L_{n,N}(a_1) L_{N,N+k} \circ L_{n,N}(b_1) \\ - L_{N,N+k}(L_{n,N}(a_1) L_{n,N}(b_1))\| < \varepsilon$$

for all $k > 0$. By the definition, $L_{n,N}(b_1) \in I_N$. Therefore $L_{N,N+k}(L_{n,N}(a_1) L_{n,N}(b_1)) \in I_N + k$. This implies that

$$\mathrm{dist}(ab, J) < \varepsilon$$

for all $\varepsilon > 0$. Hence $ab \in J$. Similarly $ba \in J$. □

Definition 3.7. Let $B_1 = C(\mathbb{T})$ and $B_{n+1} = M_{(n+1)!}(C(\mathbb{T}))$, $n = 1, 2, \ldots$ Define $h_n : B_n \to B_{n+1}$ by $h_n(b) = \mathrm{diag}(b, b(\xi_1), \ldots, b(\xi_n))$, $n = 1, 2, \ldots$ Let $B_\infty = \lim_n (B_n, h_n)$. Then B_∞ is a unital simple C^*-algebra with $\mathrm{TR}(B_\infty) = 0$

(see Definition 4.2), $K_1(B_\infty) = \mathbb{Z}$ and $K_0(B_\infty) = \mathbb{Z} \oplus \mathbb{Q}$ with $K_0(B_\infty)_+ = \{(n,r) : n > 0, r \in \mathbb{Q}_+ \setminus \{0\}\} \cup \{0\}$.

PROPOSITION 3.8. *Let $\pi_\infty : E \to E/J$ be the quotient map. Then $\pi_\infty(E) \cong B_\infty$.*

Proof. We first show that, for each n, $L_{n,\infty}(E_n) \cap J = L_{n,\infty}(I_n)$. Let $a \in E_n \setminus I_n$. Then, by the construction, for all $m > 0$,

$$\mathrm{dist}(L_{n,m}(a), I_{n+m}) \geq \|\pi_{n!}(a)\|,$$

where $\pi_{n!} : E_n \to E_n/I_n$ is the quotient map. This implies that

$$\mathrm{dist}(L_{n,\infty}(a), J) \geq \|\pi_{n!}(a)\|.$$

Therefore $L_{n,\infty}(E_n) \cap J = L_{n,\infty}(I_n)$.

Now we have

$$L_{n,\infty}(E_n)/J \cong B_n.$$

From the construction there is an isomorphism from $L_n(E_n)/I_{n+1}$ to $L_{n,\infty}(E_n)/J$. Denote by $j_n : L_{n,\infty}(E_n)/J \to L_{n+1,\infty}(E_{n+1})/J$ the map induced by L_n and by γ_n the isomorphism from $L_{n,\infty}(E_n)/J$ onto B_n. We obtain the following intertwining:

$$\begin{array}{ccc} L_{n,\infty}(E_n)/J & \xrightarrow{j_n} & L_{n+1,\infty}(E_{n+1})/J \\ \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \gamma_{n+1}} \\ B_n & \xrightarrow{h_{n,\infty}} & B_{n+1} \end{array}$$

This implies that $B_\infty \cong E/J$. □

4. Tracial topological rank of E

Through the rest of paper, we will write $f_{\delta_2}^{\delta_1}$ (where $0 < \delta_2 < \delta_1 < 1$) for the following non-negative continuous function on $[0, \infty)$:

$$f_{\delta_2}^{\delta_1}(t) = \begin{cases} 1, & t \geq \delta_1, \\ (t - \delta_2)/(\delta_1 - \delta_2), & \delta_2 < t < \delta_1, \\ 0, & t \leq \delta_2. \end{cases}$$

DEFINITION 4.1. Let a and b be two positive elements in a C^*-algebra A. We write $[a] \leq [b]$ if there exists $x \in A$ such that $a = x^*x$ and $xx^* \in \overline{bAb}$, and $[a] = [b]$ if $a = x^*x$ and $b = xx^*$. For more information on this relation, see [Cu1], [Cu2] and [HLX1].

DEFINITION 4.2 ([Ln4] and [HLX1]). Recall that a unital C^*-algebra A is said to have tracial topological rank zero if the following holds: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ containing a nonzero element $a \in A_+$, and any real numbers $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there is a projection $p \in A$ and a finite dimensional C^*-subalgebra B of A with $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$, and
(3) $[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$.

If A has tracial topological rank zero, we will write $\mathrm{TR}(A) = 0$. If A is non-unital, we say that A has tracial topological rank zero if $\mathrm{TR}(\tilde{A}) = 0$.

We will show that the C^*-algebra E constructed in the previous section has tracial topological rank zero. The proof is similar to that in [Ln7]. We will use the following two lemmas:

LEMMA 4.3 ([HLX1, Lemm 1.8]). *Let $0 < \sigma_4 < \sigma_3 < 1$. There is $\delta = \delta(\sigma_3, \sigma_4) > 0$ such that for any C^*-algebra A, any $a, b \in A_+$ with $\|a\| \leq 1$, $\|b\| \leq 1$, and any σ_1, σ_2 with $\sigma_3 < \sigma_2 < \sigma_1 < 1$, $\|a - b\| < \delta$ implies*

$$[f_{\sigma_2}^{\sigma_1}(a)] \leq [f_{\sigma_4}^{\sigma_3}(b)].$$

LEMMA 4.4 ([Ln7, Lemma 3.4]). *Let $0 < \sigma_4 < \sigma_3 < 1$. There is $\delta_1 = \delta(\sigma_3, \sigma_4) > 0$ such that for any C^*-algebra A, any $a, b \in A_+$ and $x \in A$ with $\|x\| \leq 1$, $\|a\| \leq 1$, $\|b\| \leq 1$ and any σ_1, σ_2 with $\sigma_3 < \sigma_2 < \sigma_1 < 1$, then $\|x^*x - a\| < \delta_1$ and $\|xx^* - b\| < \delta_1$ imply*

$$[f_{\sigma_2}^{\sigma_1}(a)] \leq [f_{\sigma_4}^{\sigma_3}(b)].$$

LEMMA 4.5. $\mathrm{TR}(E) = 0$.

Proof. By Definition 1.11 (see also Proposition 1.17) in [HLX1], it suffices to show the following:

For any $\varepsilon > 0$, any $0 < \sigma_2 < \sigma_1 < 1$, any finite subset $\mathcal{F}$ of E and a nonzero element $a \in E_+$, there is a projection $p \in E$ and a finite dimensional C^*-subalgebra $C \subset E$ with $1_C = p$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $\mathrm{dist}(x, C) < \varepsilon$ for all $x \in \mathcal{F}$, and
(3) $[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \leq [f_{\sigma_4}^{\sigma_3}(pap)]$ for some $0 < \sigma_4 < \sigma_3 < \sigma_2$.

Without loss generality, we may assume that $\|a\| = 1$. Fix $0 < d_2 < d_1 < \min\{1/8, \sigma_2\}$. Let $\delta(d_1, d_2) > 0$ be as in Lemma 4.4. There is an integer n such that $1/n < \varepsilon/4$, and a finite subset $S \subset E_n$ such that $\mathcal{F} \cup \{a\} \subset L_{n,\infty}(S)$. Suppose that $L_{n,\infty}(b) = a$, where $0 \leq b \leq 1$ is in E_n and $\|b\| = 1$. We may assume that $L_{n,\infty}(S') \subset L_{l,\infty}(\mathcal{F}_l)$, where $S' = \{cd : c, d \in S\}$ and $\mathcal{F}_l$ is as in Definition 3.5.

Choose a large integer $l > (n+1)^2$ such that $\max\{1/2^{l-2}, 1/l\} < \delta(d_1, d_2)/2$ and $\|\psi_l(L_{n,l-1}(b))\| \geq (1/2)\|b\|$. (Note that $1/l < \varepsilon/16$.) For $s \in S$ write (in E_l)

$$L_{n,l}(s) = \mathrm{diag}(s, L(s)), \quad \text{with} \quad L_{n,l}(1_{E_n}) = \mathrm{diag}(1_{E_n}, L(1_{E_n})),$$

where $L(s) \in C_l$. (See the construction of E.) Since $L_{l,\infty}$ is $\mathcal{F}_l$-$1/(l+1)2^l \cdot \delta_{\dim C_l}/2^l$-multiplicative, by Proposition 3.3 there is a homomorphism $h : C_l \to E$ such that

$$\|L_{l,\infty}|_{C_l} - h\| < 1/2^{l-1}.$$

Let $p' = \mathrm{diag}(0, L(1_{E_n}))$. Then $p' \in C_l$. Hence there is a projection $p \in h(C_l)$ such that $\|L_{l,\infty}(p') - p\| < \min\{1/2^{l-1}, \varepsilon/2\}$. Since $L_{l,\infty}$ is $\mathcal{F}_l$-$1/(l+1)2^l \cdot \delta_{\dim C_l}/2^l$-multiplicative, we have

(1) $\|px - xp\| < \varepsilon$ for $x \in \mathcal{F}$, and
(2) $pxp \in_\varepsilon h(C_l)$ for $x \in \mathcal{F}$.

To show (3) we consider two cases:

Case (i): $b \in (I_n)_+$. We may assume that

$$\|e_l b - b\| < \min\{\delta(d_1, d_2)/4, \varepsilon/4\}.$$

Let $b_1 = e_l b e_l$ and $b_1' = L_{n,l-1}(b_1)$. Thus, $\psi(b_1') \neq 0$. We have

$$L_{n,l}(b_1) = \mathrm{diag}(b_1, \Phi_n(b_1), \psi_l(b_1'), \dots, \psi_l(b_1')),$$

where $\Phi_n : I_n \to I_l$ is a contractive completely positive linear map and $\Phi_n(I_n)$ is contained in C_l and ψ_l is repeated l times. Note that $\|\psi(b_1')\| > 1/4$. So, $\mathrm{diag}(\psi_l(b_1'), \dots, \psi_l(b_1'))$ has an eigenvalue λ with $\lambda \geq 1/4$ and its rank in C_l' (see the construction of E) is at least l. We have

$$[b_1] \leq [e_l] \quad \text{and} \quad (1/4)[e_l] \leq [\mathrm{diag}(\psi_l(b_1'), \dots, \psi_l(b_1'))],$$

where $\psi_l(b_1')$ is repeated l times.

Put $c = \mathrm{diag}(0, \Phi_n(b_1), \psi_l(b_1'), \dots, \psi_l(b_1'))$ and $b' = \mathrm{diag}(b_1, 0, \dots, 0)$. Since $\{u_{ij}\}_{i=1}^l \subset \mathcal{F}_l$, there is $x \in \mathcal{F}_l$ such that

$$x^*x = b' \quad \text{and} \quad xx^* \in C',$$

where $C' = e_l C_l' e_l$. Moreover, c contains an eigenvalue λ with $\lambda \geq 1/4$ and the corresponding spectral projection e larger than a projection in C_l' with rank l. Therefore, there exists $v \in C_l$ such that

$$v^*v = e_l \quad \text{and} \quad vv^* \leq e.$$

Note that $f_{1/8}^{1/4}(c) \geq e$. This implies that there is $z \in C_l$ such that

$$z^*z = xx^* \quad \text{and} \quad zz^* f_{1/8}^{1/4}(c) = zz^*.$$

Let $y = L_{l,\infty}(x)$ and $b'' = pL_{l,\infty}(b')p$. Since $L_{l,\infty}$ is $\mathcal{F}_l$-$1/(l+1) \cdot 2^l \cdot \delta_{\dim C_l}/2^l$-multiplicative and $\|L_{l,\infty}|_{C_l} - h\| < 1/2^{l-1}$, we have

$$\|y^*y - b''\| < 1/2^{l-2} \quad \text{and} \quad \|yy^* - h(xx^*)\| < 1/2^{l-2}.$$

We also have

$$\|b'' - (1-p)a(1-p)\| < 1/2^{l-2} \quad \text{and} \quad \|h(c) - pap\| < 1/2^{l-2}.$$

Moreover,

$$h(z^*z) = h(xx^*) \quad \text{and} \quad h(zz^*)h(f_{1/8}^{1/4}(c)) = h(zz^*).$$

Therefore, by Lemma 4.4,

$$[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \le [f_{d_2}^{d_1}(h(xx^*))].$$

We also have that $[f_{d_2}^{d_1}(h(xx^*))] = [f_{d_2}^{d_1}(h(zz^*))]$. Therefore

$$[f_{d_2}^{d_1}(h(zz^*))] \le [h(zz^*)] \le [h(f_{1/8}^{1/4}(c))].$$

It then follows from the proof of Lemma 4.4 that there are $0 < \sigma_4 < \sigma_3 < d_2$ such that

$$[h(f_{1/8}^{1/4}(c))] \le [f_{\sigma_4}^{\sigma_3}(pap)],$$

because $\|h(c) - pap\| < 1/2^{l-2}$. Hence

$$[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \le [f_{\sigma_4}^{\sigma_3}(pap)].$$

Case (ii): $b \in (E_n)_+ \setminus I_n$. This part of the proof is just a slight modification of the proof in Case (i). Take $0 < \sigma_{10} < \sigma_9 < \cdots < \sigma_4 < \sigma_3 < d_2$. We note that $\phi_i^{(n+1)} \circ L_n(a)$ has the form

$$\operatorname{diag}(\pi_{n!}(a)(\xi_i), \phi_1^{(n)}(a), \dots, \phi_{n-1}^{(n)}(a), \pi_{n!}(a)(\xi_n))$$

for $0 < i < n$ and $a \in E_n$. Since $\{\xi_n\}$ is dense in $\mathbb{T}$, without loss of generality we may assume that $\pi_{n!}(b)(\xi_m) \neq 0$ and $n < m < m! < l$. By the construction we may write

$$L_{n,l}(b) = \operatorname{diag}(b, L'(b), \pi_{n!}(b)(\xi_m), \dots, \pi_{n!}(b)(\xi_m), L''(b)),$$

where $\pi_{n!}(b)(\xi_m)$ is repeated $m!$ times and $L'(b), L''(b) \in C_l$. Note that

$$\begin{aligned}&\operatorname{diag}(0, L'(b), \pi_{n!}(b)(\xi_m), \dots, \pi_{n!}(b)(\xi_m), L''(b))\\ &\qquad \ge \operatorname{diag}(0, 0, \dots, 0, \pi_{n!}(b)(\xi_m), \dots, \pi_{n!}(b)(\xi_m), 0).\end{aligned}$$

Since $\{u_{ij}\} \subset \mathcal{F}_l$, there is $z_k \in \mathcal{F}_l$ such that

$$z_k^* z_k = \operatorname{diag}(b, 0, 0, \dots, 0) \quad \text{and} \quad z_k z_k^* = \operatorname{diag}(0, \dots, 0, b, 0),$$

where b is in the $(k+1)$st position. We note that there is $c \in M_{l!/n!}(\mathbb{C} \cdot 1_{E_n})$ such that

$$c^*c = \operatorname{diag}(0, 1_{E_n}, 0, \dots, 0)$$

and

$$cc^* \le \operatorname{diag}(0, \dots, 0, \pi_{n!}(b)(\xi_m), \dots, \pi_{n!}(b)(\xi_m), 0).$$

Indeed, since $\pi_{n!}(b)(\xi_m) \neq 0$, $\pi_{n!}(b)(\xi_m)$ is unitarily equivalent to $\operatorname{diag}(\alpha_1, \dots, \alpha_{n!})$ for some $\alpha_1, \dots, \alpha_{n!} \in \mathbb{C}$ with $\alpha_1 \neq 0$. Since t$\pi_{n!}(b)(\xi_m)$ is repeated $m!$ times and $m > n$, we have

$$[\operatorname{diag}(0, 1_{E_n}, 0, \dots, 0)] \le [\operatorname{diag}(0, \dots, 0, \pi_{n!}(b)(\xi_m), \dots, \pi_{n!}(b)(\xi_m), 0)].$$

Note also $\mathrm{diag}(0, b, 0, \ldots, 0) \le \mathrm{diag}(0, 1_{E_n}, 0, \dot{s}, 0)$ and $(L_{l,\infty})|_{M_{l!}(\mathbb{C}\cdot 1_{E_1})}$ is a homomorphism. Therefore we have

$$\begin{aligned}
[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] &\le [f_{d_2}^{d_1}(L_{l,\infty}(z_1^* z_1))] \\
&= [f_{d_2}^{d_1}(L_{l,\infty}(z_1 z_1^*))] \\
&\le [f_{\sigma_4}^{\sigma_3}(L_{l,\infty}(c^* c)] \\
&\le [f_{\sigma_4}^{\sigma_3}(L_{l,\infty}(cc^*))] \\
&\le [f_{\sigma_6}^{\sigma_5}(L_{l,\infty}(\mathrm{diag}(0, \pi_{n!}(b)(\xi_m), \ldots, \pi_{n!}(b)(\xi_m), 0))] \\
&\le [f_{\sigma_8}^{\sigma_7}(\mathrm{diag}(0, L'(b), \pi_{n!}(b)(\xi_m), \ldots, \pi_{n!}(b)(\xi_m), L''(b))] \\
&\le [f_{\sigma_{10}}^{\sigma_9}(pap)].
\end{aligned}$$

and hence

$$[f_{\sigma_2}^{\sigma_1}((1-p)a(1-p))] \le [f_{\sigma_{10}}^{\sigma_9}(pap)].$$

This shows that $\mathrm{TR}(E) = 0$. □

Corollary 4.6. *J is an AF algebra.*

Proof. Given a finite subset $\mathcal{F} \subset J$, we may assume that there is a finite subset $\mathcal{G} \subset I_n$ such that $L_{n,\infty}(\mathcal{G}) = \mathcal{F}$. Furthermore, we may assume that $\mathcal{G} \subset S_{k(n)} \otimes M_{n!}$ with $k(n) > n$. Since $C_l'' \oplus C_{l+1}' \subset C_{l+1}''$ for $1 \le l$ and

$$L_{n,k(n)}(S_{k(n)} \otimes M_{n!}) \subset S_{k(n)} \otimes M_{n!} \oplus C_n' \oplus C_{n+1}' \oplus \cdots \oplus C_{k(n)-1}',$$

it follows that $L_{n,k(n)}(S_{k(n)} \otimes M_{n!}) \subset C_{k(n)}''$. By the choice of $\mathcal{S}_{k(n)}$ and $\eta_{k(n)}$, we see that there is a homomorphism $\bar{h}_{k(n)} : C_{k(n)}'' \to E$ such that

$$\|L_{k(n),\infty}|_{C_{k(n)}''} - \bar{h}_{k(n)}\| < 1/2^{k(n)}.$$

Let $B_0 = \bar{h}_{k(n)}(C_{k(n)}'')$. Then B_0 is a finite dimensional C^*-algebra and for any $x \in \mathcal{G}$ we have

$$\begin{aligned}
&\|L_{n,\infty}(x) - \bar{h}_{k(n)}(L_{n,k(n)}(x))\| \\
&\quad \le \|L_{n,\infty}(x) - L_{k(n),\infty}(L_{n,k(n)}(x))\| \\
&\qquad\quad + \|L_{k(n),\infty}(L_{n,k(n)}(x)) - \bar{h}_{k(n)}(L_{n,k(n)}(x))\| \\
&\quad < 1/2^{k(n)}.
\end{aligned}$$

This implies that

$$\mathrm{dist}(\mathcal{F}, B_0) < 1/2^{k(n)}.$$

From this one sees that J is an AF-algebra. □

5. Tracially quasidiagonal extensions

Definition 5.1. Let

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0,$$

be a short exact sequence of C^*-algebras. In [HLX2], we say that (E, I) is *tracially quasidiagonal* if, for any $\varepsilon > 0$, any nonzero $a \in E_+$, any finite subset $\mathcal{F} \subset E$ and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exists a C^*-subalgebra $D \subset E$ with $1_D = p$ such that

(1) $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon D$ for all $x \in \mathcal{F}$,
(3) $D \cap I = pIp$ and $(D, D \cap I)$ is quasidiagonal, and
(4) $[f_{\sigma_4}^{\sigma_3}((1-p)a(1-p))] \le [f_{\sigma_2}^{\sigma_1}(pap)]$.

In [HLX2] we showed that if $\mathrm{TR}(I) = 0 = \mathrm{TR}(A)$ then $\mathrm{TR}(E) = 0$ if (E, I) is tracially quasidiagonal. Moreover, if $\mathrm{TR}(eIe) = 0$ for every projection $e \in E$, then $\mathrm{TR}(E) = 0$ also implies that the extension is tracially quasidiagonal.

It is clear that if (E, I) is quasidiagonal, then (E, I) is tracially quasidiagonal. (Take $p = 1$.) On the other hand, Corollary 5.5 below says that there are tracially quasidiagonal extensions that are not quasidiagonal.

Theorem 5.2. *The extension*

$$0 \longrightarrow J \longrightarrow E \longrightarrow B_\infty \longrightarrow 0,$$

is tracially quasidiagonal.

Proof. Since $\mathrm{TR}(B_\infty) = \mathrm{TR}(E) = \mathrm{TR}(J) = 0$, and $\mathrm{TR}(eJe) = 0$ for every projection $e \in E$ by Lemma 4.5 and Corollary 4.6, it follows from [HLX2] that the extension is tracially quasidiagonal. ☐

Lemma 5.3. *Let A_n be a sequence of unital C^*-algebras and $A = \lim_{n\to\infty} (A_n, \varphi_{n,m})$ be a generalized inductive limit in the sense of* [BE]. *Suppose that each $\varphi_{n,m} : A_n \to A_m$ is unital with $\|\varphi_n(a)\| = \|\varphi_{n,m}(a)\|$ for $a \in A_n$ and $m > n$. Let $u \in A$ be a unitary. Then for any $\varepsilon > 0$ there is $n > 0$ and a unitary $v \in A_n$ such that*

$$\|\varphi_n(v) - u\| < \varepsilon,$$

where φ_n is an induced map by the inductive limit from A_n into A.

Proof. By definition there is a sequence $\{\varphi_{n_k}(a_k)\}$, where $a_k \in A_{n_k}$ that converges to u. Therefore, we may assume that

$$\|\varphi_{n_k}(a_k) - u\| < 1/2^{k+2}.$$

and

$$\|\varphi_{n_k}(a_k^*)\varphi_{n_k}(a_k) - 1\| < 1/2^{k+2}, \quad \|\varphi_{n_k}(a_k)\varphi_{n_k}(a_k^*) - 1\| < 1/2^{k+2}.$$

From the definition of a generalized inductive limit it follows that there is $m_k > n_k$ such that for any $x, y \in A_{n_k}$

$$\|\varphi_{m_k}(\varphi_{n_k,m_k}(x)+\varphi_{n_k,m_k}(y))-(\varphi_{n_k}(x)+\varphi_{n_k}(y))\| < 1/2^{k+2},$$
$$\|\varphi_{m_k}(\varphi_{n_k,m_k}(x)^*)-\varphi_{n_k}(x)^*\| < 1/2^{k+2},$$
$$\|\varphi_{m_k}(\varphi_{n_k,m_k}(x)\varphi_{n_k,m_k}(y))-\varphi_{n_k}(x)\varphi_{n_k}(y)\| < 1/2^{k+2}.$$

Set $b_k = \varphi_{n_k,m_k}(a_k) \in A_{m_k}$. Since

$$\begin{aligned}\|\phi_{n_k}(a_k^*a_k)-1\| &= \|\phi_{n_k}(a_k^*a_k)-\phi_{m_k}(\phi_{n_k,m_k}(a_k^*)\phi_{n_k,m_k}(a_k))\\ &\qquad + \phi_{m_k}(\phi_{n_k,m_k}(a_k^*)\phi_{n_k,m_k}(a_k))-1\|\\ &\le \|\phi_{n_k}(a_k^*a_k)-\phi_{m_k}(\phi_{n_k,m_k}(a_k^*)\phi_{n_k,m_k}(a_k))\|\\ &\qquad + \|\phi_{m_k}(\phi_{n_k,m_k}(a_k^*)\phi_{n_k,m_k}(a_k))-1\|\\ &\le 1/2^{k+2} + \|\phi_{m_k}(\phi_{n_k,m_k}(a_k^*)\phi_{n_k,m_k}(a_k))\\ &\qquad - \phi_{n_k}(a_k^*)\phi_{n_k}(a_k)\| + \|\phi_{n_k}(a_k^*)\phi_{n_k}(a_k)-1\|\\ &\le 3/2^{k+2},\end{aligned}$$

we have

$$\begin{aligned}\|\varphi_{m_k}(b_k^*b_k-1)\| &\le \|\varphi_{m_k}(b_k^*b_k-1)-(\varphi_{n_k}(a_k^*a_k)-1)\|\\ &\qquad + \|\varphi_{n_k}(a_k^*a_k)-1\|\\ &< 1/2^{k+2}+3/2^{k+2} = 1/2^k,\\ \|\varphi_{m_k}(b_kb_k^*-1)\| &< 1/2^k\end{aligned}$$

and

$$\begin{aligned}\|\varphi_k(b_k)-u\| &= \|\varphi_{m_k}(\varphi_{n_k,m_k}(a_k))-u\|\\ &\le \|\varphi_{m_k}(\varphi_{n_k,m_k}(a_k))-\varphi_{n_k}(a_k)\| + \|\varphi_{n_k}(a_k)-u\|\\ &< 1/2^{k+1}.\end{aligned}$$

Hence b_k is an invertible element in A_{m_k}. Set $v = b_k|b_k|^{-1}$. Then v is a unitary, and the distance between v and b_k is small (depending on k). Hence, taking a sufficient large k, we have a unitary $v \in A_{m_k}$ such that

$$\|\varphi_{m_k}(v)-u\| < \varepsilon. \qquad \square$$

THEOREM 5.4. *A C^*-algebra E has topological stable rank two.*

Proof. Take a unitary $u \in B_\infty$ such that $0 \neq [u]_1 \in K_1(B_\infty)$. Suppose that u can be lifted to a unitary $\tilde{u}$ in E. We will get a contradiction.

Since a system $(E_n, L_{n,m})$ is a generalized inductive limit, by Lemma 5.3 there exists $n \in \mathbb{N}$ and a unitary $v \in E_n$ such that

$$\|L_{n,\infty}(v)-\tilde{u}\| < 1.$$

From the commutative diagram

$$\begin{array}{ccc} E_n & \xrightarrow{L_{n,\infty}} & E \\ \Big\downarrow{\scriptstyle \pi_{n!}} & & \Big\downarrow{\scriptstyle \pi_\infty} \\ B_n & \xrightarrow{h_{n,\infty}} & B_\infty \end{array}$$

we conclude that $w_n = \pi_{n!}(v)$ is a unitary in B_n and $h_{n,\infty}(w_n) = \pi_\infty \circ L_{n,\infty}(v)$. Set $w = h_{n,\infty}(w_n)$. Then $w \in B_\infty$ is a unitary, and

$$\begin{aligned} \|w - u\| &= \|\pi_\infty \circ L_{n,\infty}(v) - \pi_\infty(\tilde{u})\| \\ &\le \|L_{n,\infty}(v) - \tilde{u}\| < 1. \end{aligned}$$

Hence, $[w]_1 = [u]_1$ in $K_1(B_\infty)$. Therefore, $[w_n]_1 \neq 0$ in $K_1(B_n)$, because the induced map $(h_{n,\infty})_* : K_1(B_n) \to K_1(B_\infty)$ is an identity (see Definition 3.7).

On the other hand, from the construction we have $0 \neq \partial_1([w_n]_1) \in K_0(I_n)$. But this gives a contradiction, for

$$\partial_1([w_n]_1) = \partial_1 \circ \pi_*([v]_1) = 0.$$

So by [Ni, Lemma 3] or [LR, Proposition 4] E has topological stable rank more than one. Hence by [Rf, Corollary 4.2] we conclude that E has topological stable rank two. □

Corollary 5.5. *The extension*

$$0 \longrightarrow J \longrightarrow E \longrightarrow B_\infty \longrightarrow 0$$

is not quasidiagonal.

Proof. From the previous theorem it follows that the index map $\partial_1 : K_1(B_\infty) \to K_0(J)$ is non-zero. Hence, by [BrD, Theorem 8] the extension is not quasidiagonal. □

Remark 5.6. (1) The C^*-algebra E is QD in [HLX3, Theorem 4.6].

(2) A C^*-algebra E in (1) is not an AH-algebra, that is, it can not be written as the inductive limit of direct sums of homogeneous C^*-algebras. Indeed, since J and B_∞ have real rank zero (see [G, Theorem 9]), if E is an AH-algebra, the extension has to be quasidiagonal by [BrD, Proposition 11]. This is a contradiction to Corollary 5.5.

(3) In [Ln7] the first author constructed an example of a unital C^*-algebra A which has tracial topological rank zero, but real rank greater than zero. This also gives an extension $0 \to J \to A \to B \to 0$ with a non-zero index map $\partial_0 : K_0(B) \to K_1(I)$. In the present note we have constructed another tracially quasidiagonal extension in Corollary 5.5 with a non-zero index map $\partial_1 : K_1(B) \to K_0(I)$. This implies that a more complicated index is needed to characterize tracially quasidiagonal extensions.

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