Analytical Mechanics

Shinichi Hirai

Dept. Robotics, Ritsumeikan Univ.

Agenda

1. Schedule
2. Introduction to Analytical Mechanics
3. Illustrative Examples
   - Free fall of a mass
   - Open/Closed link mechanisms
   - Watt's governor
   - Beam deformation
4. MATLAB environment
5. Summary

Schedule (tentative)

Introduction	1 week
Variational Principles	2 weeks
MATLAB	2 weeks
Link Mechanisms	2 weeks
Rigid Body Rotation	2 weeks
Elastic Deformation	4 weeks
Inelastic Deformation	2 weeks

web page

http://www.ritsumei.ac.jp/~hirai/
English ⟶ Classes ⟶ 2024 Analytical Mechanics

or directly
http://www.ritsumei.ac.jp/~hirai/edu/2024/analytical_mechanics/analytical_mechanics.html

Newton mechanics vs Lagrange mechanics

Newton mechanics

vectors
linear momentum, force, angular momentum, moment, ⋯
vectors depend on coordinate systems

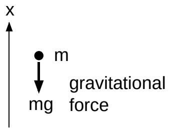
internal forces have to be identified and eliminated
constraints should be solved explicitly

Lagrange mechanics

scalars
kinetic energy, potential energy, work done by external forces, ⋯
scalars are independent of coordinate systems

internal forces do not appear in Lagrangian
constraints can be incorporated into Lagrangian

Free fall of a mass (Newton mechanics)



Newton mechanics

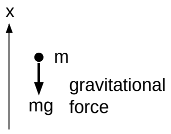
linear momentum $p = mv$

Newton's eq. of motion $\frac{\mathrm{d}p}{\mathrm{d}t} = -mg$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = m\dot{v}$$

differential equation $m\dot{v} = -mg$

Free fall of a mass (Lagrange mechanics)



Lagrange mechanics

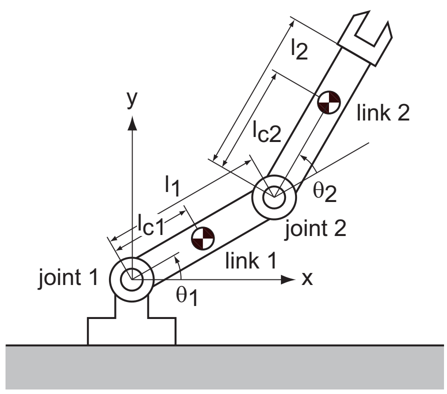
kinetic energy $T = \frac{1}{2}mv^2$

potential energy $U = mgx$

Lagrangian $\mathcal{L} = T - U = \frac{1}{2}mv^2 - mgx$

Lagrange eq. of motion $\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial v}\right) = -mg - m\dot{v} = 0$

Open link mechanism



Open link mechanism

Newton mechanics

1. identify all forces applied to each link (inc. internal forces)
2. apply Newton's eqs. of motion (and Euler's eqs. of rotation)

$$m_1\dot{\boldsymbol{v}}_1 = m_1\boldsymbol{g} + \boldsymbol{R}^{1,0} + \boldsymbol{R}^{1,2}, \quad m_2\dot{\boldsymbol{v}}_2 = m_2\boldsymbol{g} + \boldsymbol{R}^{2,1}, \quad \cdots$$

3. eliminate internal forces $\boldsymbol{R}^{1,0}$, $\boldsymbol{R}^{1,2}$, $\boldsymbol{R}^{2,1}$

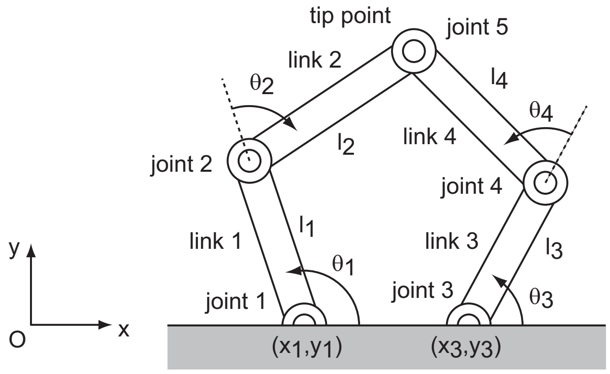
Lagrange mechanics

1. formulate kinetic and potential energies

$$T = T_1 + T_2, \quad U = U_1 + U_2$$

2. apply Lagrange's eqs. of motion to Lagrangian $\mathcal{L} = T - U$

Closed link mechanism



Closed link mechanism

left arm link 1 – link 2 ⇒ open link mech. ⇒ Lagrangian $\mathcal{L}_{\rm left}$
right arm link 3 – link 4 ⇒ open link mech. ⇒ Lagrangian $\mathcal{L}_{\rm right}$

geometric constraints

tip position of left arm = tip position of right arm

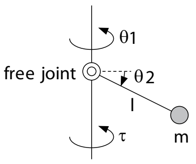
$$X \triangleq l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3 = 0$$

$$Y \triangleq l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3 = 0$$

Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm left} + \mathcal{L}_{\rm right} + \lambda_x X + \lambda_y Y$$

Watt's governor (Newton mechanics)



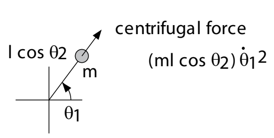
rotation around driving axis

$$I_1 = m(l\cos\theta_2)^2 = ml^2\cos^2\theta_2$$

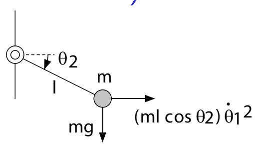
$$\tau = \frac{\rm d}{{\rm d}t}(I_1\dot\theta_1) = \dot I_1\dot\theta_1 + I_1\ddot\theta_1$$

$$\tau = \left\{ml^2\cdot 2\cos\theta_2(-\sin\theta_2)\dot\theta_2\right\}\dot\theta_1 + \left\{ml^2\cos^2\theta_2\right\}\ddot\theta_1$$

Watt's governor (Newton mechanics)



top view



side view

rotation around free-joint axis

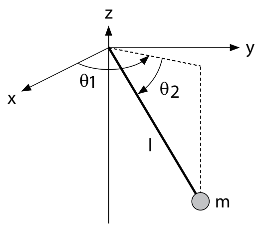
$$I_2 = ml^2$$

$$\frac{\rm d}{{\rm d}t}(I_2\dot\theta_2) = mg\times l\cos\theta_2 - ml\cos\theta_2\dot\theta_1^2\times l\sin\theta_2$$

$$ml^2\ddot\theta_2 = mgl\cos\theta_2 - ml^2\cos\theta_2\sin\theta_2\,\dot\theta_1^{\,2}$$

need to identify centrifugal force

Watt's governor (Lagrange mechanics)



position of mass

$$\boldsymbol{x} = \begin{bmatrix} l\cos\theta_1\cos\theta_2 \\ l\sin\theta_1\cos\theta_2 \\ -l\sin\theta_2 \end{bmatrix} = l\begin{bmatrix} \cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 \\ -\sin\theta_2 \end{bmatrix}$$

Watt's governor (Lagrange mechanics)

velocity of mass

$$\boldsymbol{v} = \frac{{\rm d}\boldsymbol{x}}{{\rm d}t} = \frac{\partial\boldsymbol{x}}{\partial\theta_1}\frac{{\rm d}\theta_1}{{\rm d}t} + \frac{\partial\boldsymbol{x}}{\partial\theta_2}\frac{{\rm d}\theta_2}{{\rm d}t}$$

$$= l\dot\theta_1\begin{bmatrix} -\sin\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2 \\ 0 \end{bmatrix} + l\dot\theta_2\begin{bmatrix} -\cos\theta_1\sin\theta_2 \\ -\sin\theta_1\sin\theta_2 \\ -\cos\theta_2 \end{bmatrix}$$

$$v^2 = (l\dot\theta_1)^2\cdot\cos^2\theta_2 + (l\dot\theta_2)^2\cdot 1 + 2(l\dot\theta_1)(l\dot\theta_2)\cdot 0$$

$$= l^2(\cos^2\theta_2\dot\theta_1^2 + \dot\theta_2^2)$$

kinetic/potential energies, work done by external torque

$$T = \frac{1}{2}ml^2(\cos\theta_2^2\dot\theta_1^2 + \dot\theta_2^2), \quad U = -mgl\sin\theta_2, \quad W = \tau\theta_1$$

Watt's governor (Lagrange mechanics)

Lagrangian

$$\mathcal{L} \triangleq T - U + W$$

Lagrange eqs. of motion

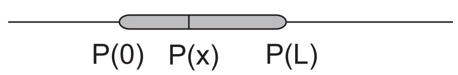
$$\frac{\partial\mathcal{L}}{\partial\theta_k} - \frac{\rm d}{{\rm d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta_k}\right) = 0, \quad (k = 1, 2)$$

$$\tau - \left\{ml^2\cdot 2\cos\theta_2(-\sin\theta_2)\dot\theta_2\right\}\dot\theta_1 - \left\{ml^2\cos^2\theta_2\right\}\ddot\theta_1 = 0$$

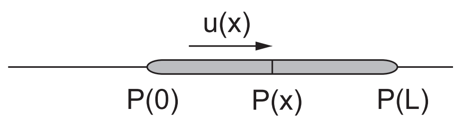
$$-ml^2\cos\theta_2\sin\theta_2\,\dot\theta_1^{\,2} + mgl\cos\theta_2 - ml^2\ddot\theta_2 = 0$$

centrifugal or Coriolis terms yield naturally

Beam deformation



natural shape



deformed shape

Deformation is described by function $u(x)$ $(0 \le x \le L)$

Beam deformation

elastic potential energy

$$U = \int_0^L \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x$$

piecewise linear approximation

dividing interval $[\,0,\ L\,]$ into 6 regions:

$$\int_0^L \mathrm{d}x = \int_{x_0}^{x_1} \mathrm{d}x + \int_{x_1}^{x_2} \mathrm{d}x + \cdots + \int_{x_5}^{x_6} \mathrm{d}x$$

linear approximation:

$$\int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x \approx \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

Beam deformation

elastic potential energy

$$U = \frac{1}{2} \begin{bmatrix} u_0 & u_1 & \cdots & u_6 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_5 \\ u_6 \end{bmatrix}$$

Deformation is described by a finite number of variables u_0 through u_6

finite element method (FEM)

What is MATLAB?

1. Software for numerical calculation
2. can handle vectors or matrices directly
3. Functions such as ODE solvers and optimization
4. Toolboxes for various applications
5. both programming and interactive calculation

What is MATLAB?

MATLAB environment

MATLAB Total Academic Headcount (TAH)
MATLAB with all toolboxes is available

Information

`https://it.support.ritsumei.ac.jp/hc/ja`

What is MATLAB?

- Install MATLAB into your own PC or mobile
- Sample programs are on the web of the class
- You can use your own PC or mobile in class

Summary: pros & cons of Lagrange mechanics

Pros

- scalar description
- once energies and works are formulated, derivative calculation yields equations of motion directly
- do not have to introduce internal forces
- effective for complex systems, such as link mechanisms, rotating or deforming objects

Cons

- difficult to understand the derived equation intuitively
- all non-potential forces, such as friction and viscous forces, are treated as external forces