Analytical Mechanics: Link Mechanisms

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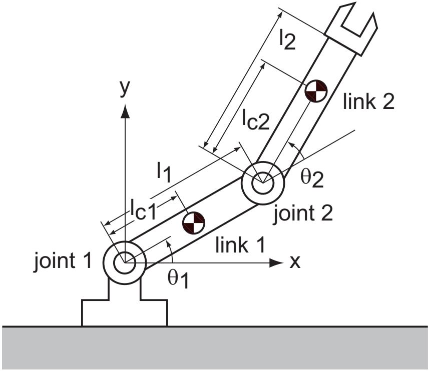
Agenda

1. Open Link Mechanism
   - Kinematics of Open Link Mechanism
   - Dynamics of 2DOF open link mechanism
2. Closed Link Mechanism
   - Kinematics of Closed Link Mechanism
   - Dynamics of 2DOF closed link mechanism

Kinematics of 2DOF open link mechanism



two link open link mechanism

l_i	length of link i
l_{ci}	distance btw. joint i and the center of mass of link i
m_i	mass of link i
J_i	inertia of moment of link i around its center of mass
θ_1	rotation angle of joint 1
θ_2	rotation angle of joint 2

Kinematics of 2DOF open link mechanism

position of the center of mass of link 1:

$$\boldsymbol{x}_{c1} \triangleq \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = l_{c1} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix}$$

position of the center of mass of link 2:

$$\boldsymbol{x}_{c2} \triangleq \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_{c2} \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

orientation angle of link 1:

$$\theta_1$$

orientation angle of link 2:

$$\theta_1 + \theta_2$$

Kinetic energy

velocity of the center of mass of link 1:

$$\dot{\boldsymbol{x}}_{c1} = l_{c1}\dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}$$

angular velocity of link 1:

$$\dot{\theta}_1$$

kinetic energy of link 1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \dot{\boldsymbol{x}}_{c1}^{\top} \dot{\boldsymbol{x}}_{c1} + \frac{1}{2} J_1 \dot{\theta}_1^2 \\ &= \frac{1}{2} (m_1 l_{c1}^2 + J_1) \dot{\theta}_1^2 \end{aligned}$$

Kinetic energy

velocity of the center of mass of link 2:

$$\dot{\boldsymbol{x}}_{c2} = l_1 \dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix} + l_{c2}(\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

angular velocity of link 2:

$$\dot{\theta}_1 + \dot{\theta}_2$$

kinetic energy of link 2:

$$\begin{aligned} T_2 =& \frac{1}{2} m_2 \dot{\boldsymbol{x}}_{c2}^\top \dot{\boldsymbol{x}}_{c2} + \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ =& \frac{1}{2} m_2 \{ l_1^2 \dot{\theta}_1^2 + l_{c2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_{c2} C_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \} + \\ & \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned}$$

Kinetic energy

total kinetic energy

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

where

$$H_{11} = J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} C_2)$$
$$H_{22} = J_2 + m_2 l_{c2}^2$$
$$H_{12} = H_{21} = J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2)$$

inertia matrix

$$H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Partial derivatives

H_{11} and $H_{12} = H_{21}$ depend on θ_2:

$$\frac{\partial H_{11}}{\partial \theta_2} = -2h_{12}, \quad \frac{\partial H_{12}}{\partial \theta_2} = \frac{\partial H_{21}}{\partial \theta_2} = -h_{12} \quad (h_{12} \triangleq m_2 l_1 l_{c2} S_2)$$

$$\dot{H}_{11} = -2h_{12}\dot{\theta}_2, \quad \dot{H}_{12} = \dot{H}_{21} = -h_{12}\dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = H_{11}\dot{\theta}_1 + H_{12}\dot{\theta}_2, \quad \frac{\partial T}{\partial \dot{\theta}_2} = H_{21}\dot{\theta}_1 + H_{22}\dot{\theta}_2$$

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_1} &= -\dot{H}_{11}\dot{\theta}_1 - H_{11}\ddot{\theta}_1 - \dot{H}_{12}\dot{\theta}_2 - H_{12}\ddot{\theta}_2 \\ &= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \\ -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_2} &= -\dot{H}_{21}\dot{\theta}_1 - H_{21}\ddot{\theta}_1 - \dot{H}_{22}\dot{\theta}_2 - H_{22}\ddot{\theta}_2 \\ &= h_{12}\dot{\theta}_1\dot{\theta}_2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2 \end{aligned}$$

Partial derivatives

H_{11}, H_{22}, and $H_{12} = H_{21}$ are independent of θ_1

$$\frac{\partial T}{\partial \theta_1} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = 0$$

H_{11} and $H_{12} = H_{21}$ depend on θ_2

$$\frac{\partial T}{\partial \theta_2} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -2h_{12} & -h_{12} \\ -h_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = -h_{12}\dot{\theta}_1^2 - h_{12}\dot{\theta}_1\dot{\theta}_2$$

contribution of kinetic energy:

$$\begin{aligned} \frac{\partial T}{\partial \theta_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_1} &= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \\ \frac{\partial T}{\partial \theta_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\theta}_2} &= -h_{12}\dot{\theta}_1^2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2 \end{aligned}$$

Gravitational potential energy

gravitational acceleration vector:

$$\boldsymbol{g} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

potential energies of link 1 and 2:

$$U_1 = -m_1 \boldsymbol{g}^\top \boldsymbol{x}_{c1}, \qquad U_2 = -m_2 \boldsymbol{g}^\top \boldsymbol{x}_{c2}$$

potential energy:

$$U = U_1 + U_2$$

contribution of potential energy:

$$-\frac{\partial U}{\partial \theta_1} = G_1 + G_2, \qquad -\frac{\partial U}{\partial \theta_2} = G_2$$

where

$$G_1 = (m_1 l_{c1} + m_2 l_1) \boldsymbol{g}^\top \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}, \quad G_2 = m_2 l_{c2} \boldsymbol{g}^\top \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

Work done by actuator torques

work done by τ_1 applied to rotational joint 1:

$$\tau_1 \theta_1$$

work done by τ_2 applied to rotational joint 2:

$$\tau_2 \theta_2$$

work done by the two actuator torques:

$$W = \tau_1 \theta_1 + \tau_2 \theta_2$$

contribution of work:

$$\frac{\partial W}{\partial \theta_1} = \tau_1, \qquad \frac{\partial W}{\partial \theta_2} = \tau_2$$

Lagrange equations of motion

Lagrangian:

$$\mathcal{L} = T - U + W$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = 0$$

let $\omega_1 \triangleq \dot{\theta}_1$ and $\omega_2 \triangleq \dot{\theta}_2$:

$$-H_{11}\dot{\omega}_1 - H_{12}\dot{\omega}_2 + h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 + \tau_1 = 0$$
$$-H_{22}\dot{\omega}_2 - H_{12}\dot{\omega}_1 - h_{12}\omega_1^2 + G_2 + \tau_2 = 0$$

Lagrange equations of motion

canonical form of ordinary differential equations:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 + \tau_1 \\ -h_{12}\omega_1^2 + G_2 + \tau_2 \end{bmatrix}$$

state variables: joint angles θ_1, θ_2 and angular velocities ω_1, ω_2

the inertia matrix is regular $\longrightarrow$ 2nd eq. is solvable

$\longrightarrow$ we can compute $\dot{\omega}_1$ and $\dot{\omega}_2$

$\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$ are functions of θ_1, θ_2, ω_1, ω_2

$$\Downarrow$$

we can sketch θ_1, θ_2, ω_1, ω_2 using an ODE solver.

Sample Programs

- class **Link**
- class **Link_Cylinder**
- class **Open_Mechanism_Two_DOF**
- class **Closed_Mechanism_Two_DOF**

class **Link_Cylinder** is a subclass of class **Link**

Sample Programs

file Link.m

```
classdef Link
    properties
        length;
        length_center;
        mass;
        inertia_of_moment_center;
        inertia_of_moment;
    end
    methods
        function obj = Link (l, lc, m, Jc, J)
            obj.length = l;
            obj.length_center = lc;
            obj.mass = m;
            obj.inertia_of_moment_center = Jc;
            obj.inertia_of_moment = J;
```

Sample Programs

Sentence

```
>> link1 = Link(2, 1, 0.0157, 0.0052, 0.0210)
```

builds a link with $l = 2$, $l_c = 1$, $m = 0.0157$, $J_c = 0.0052$, and $J = 0.0210$.

```
>> link1
link1 =
  Link properties:
                      length: 2
               length_center: 1
                        mass: 0.0157
    inertia_of_moment_center: 0.0052
           inertia_of_moment: 0.0210
>>
```

Sample Programs

building two cylindrical links of length 2, radius 0.05, and density 1

```
len = 2.00; radius = 0.05; density = 1;

len_c = len/2;
m = density * len * (pi*(radius)^2);
Jc = (1/12) * m * (3*radius^2 + len^2);
J = Jc + m * (len - len_c)^2;

link1 = Link (len, len_c, m, Jc, J);
link2 = Link (len, len_c, m, Jc, J);
```

```
>> link1
link1 =
  Link properties:
                    length: 2
             length_center: 1
                      mass: 0.0157
```

Sample Programs

building two cylindrical links of length 2, radius 0.05, and density 1

```
len = 2.00; radius = 0.05; density = 1;

link1 = Link_Cylinder (len, radius, density);
link2 = Link_Cylinder (len, radius, density);
```

```
>> link1
link1 =
  Link_Cylinder properties:
                     radius: 0.0500
                    density: 1
                     length: 2
              length_center: 1
                       mass: 0.0157
    inertia_of_moment_center: 0.0052
          inertia_of_moment: 0.0210
>>
```

Sample Programs

building an open mechanism consisting of two links

```
base = [0; 0];
grav = [0; -9.8];
robot = Open_Mechanism_Two_DOF (link1, link2, base, grav)
```

```
>> robot
robot =
  Open_Mechanism_Two_DOF properties:
            link1: [1 × 1 Link_Cylinder]
            link2: [1 × 1 Link_Cylinder]
    base_position: [2 × 1 double]
          gravity: [2 × 1 double]
           theta1: []
           theta2: []
           omega1: []
           omega2: []
               C1: []
```

Sample Programs

setting joint angles and angular velocities

```
theta = [ pi/3; pi/6 ];
omega = [ 0; 0 ];
robot = robot.joint_angles (theta, omega);
```

```
>> robot
robot =
  Open_Mechanism_Two_DOF properties:
           link1: [1 × 1 Link_Cylinder]
           link2: [1 × 1 Link_Cylinder]
   base_position: [2 × 1 double]
         gravity: [2 × 1 double]
          theta1: 1.0472
          theta2: 0.5236
          omega1: 0
          omega2: 0
              C1: 0.5000
```

Sample Programs

calculating inertia matrix and torque vector

```
[ mat, vec ] = robot.inertia_matrix_and_torque_vector
```

$$\texttt{mat} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \texttt{vec} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 \\ -h_{12}\omega_1^2 + G_2 \end{bmatrix}$$

Note `vec` does not include τ_1 or τ_2.

Solving

$$\texttt{mat}\,\dot{\boldsymbol{\omega}} = \texttt{vec} + \boldsymbol{\tau}$$

where $\boldsymbol{\tau} = [\,\tau_1,\,\tau_2\,]^\top$, yields angular acceleration $\dot{\boldsymbol{\omega}}$.

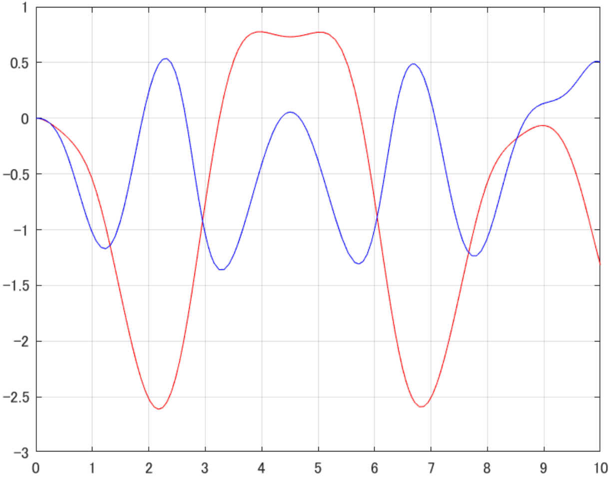
Driving by external torques

Sample Programs

- open_mechanism_2DOF_external_torques.m
  2DOF open mechanism driven by external torques
- open_mechanism_2DOF_external_torques_params.m
  equation of motion

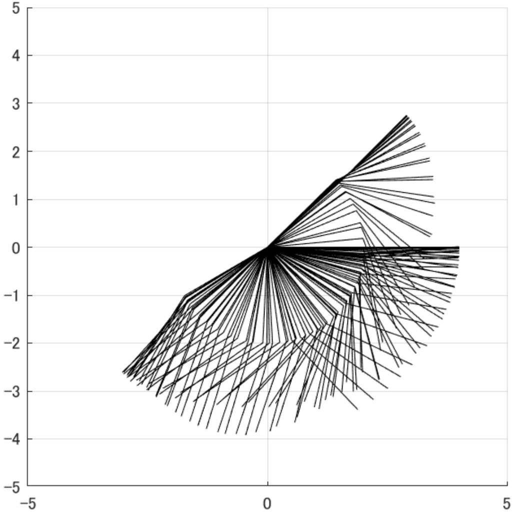
Driving by external torques

Result



Driving by external torques

Result



PD control

$$\tau_1 = -K_{\mathrm{P}1}(\theta_1 - \theta_1^d) - K_{\mathrm{D}1}\dot{\theta}_1$$
$$\tau_2 = -K_{\mathrm{P}2}(\theta_2 - \theta_2^d) - K_{\mathrm{D}2}\dot{\theta}_2$$

$$\Downarrow$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$
$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \cdots - K_{\mathrm{P}1}(\theta_1 - \theta_1^d) - K_{\mathrm{D}1}\omega_1 \\ \cdots - K_{\mathrm{P}2}(\theta_2 - \theta_2^d) - K_{\mathrm{D}2}\omega_2 \end{bmatrix}$$

current values of θ_1, θ_2, ω_1, ω_2

$$\Downarrow$$

their time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$

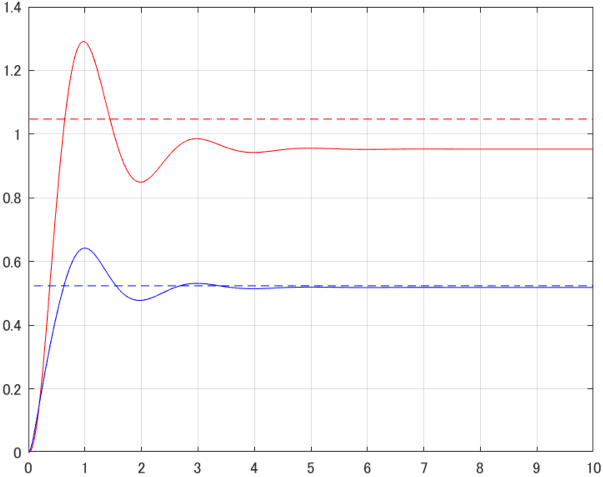
PD control

Sample Programs

- open_mechanism_2DOF_PD.m
  PD control of 2DOF open mechanism
- open_mechanism_2DOF_PD_params.m
  equation of motion

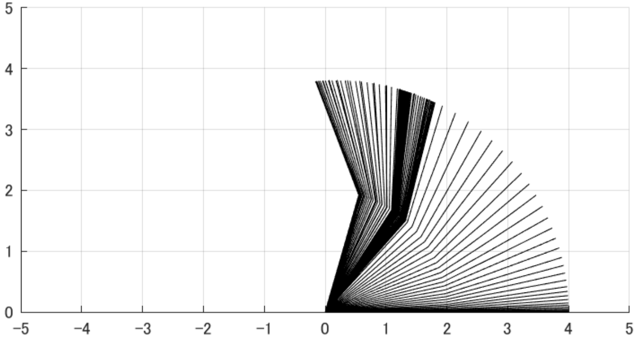
PD control

Result



PD control

Result



PI control

$$\tau_1 = -K_{\mathrm{P1}}(\theta_1 - \theta_1^d) - K_{\mathrm{I1}} \int_0^t \{(\theta_1 - \theta_1^d(\tau))\} \, \mathrm{d}\tau$$

$$\tau_2 = -K_{\mathrm{P2}}(\theta_2 - \theta_2^d) - K_{\mathrm{I2}} \int_0^t \{(\theta_2 - \theta_2^d(\tau))\} \, \mathrm{d}\tau$$

Introduce additional variables:

$$\xi_1 \stackrel{\triangle}{=} \int_0^t \{(\theta_1 - \theta_1^d(\tau))\} \, \mathrm{d}\tau$$

$$\xi_2 \stackrel{\triangle}{=} \int_0^t \{(\theta_2 - \theta_2^d(\tau))\} \, \mathrm{d}\tau$$

$$\dot{\xi}_1 = \theta_1 - \theta_1^d, \quad \tau_1 = -K_{\mathrm{P1}}(\theta_1 - \theta_1^d) - K_{\mathrm{I1}}\xi_1$$

$$\dot{\xi}_2 = \theta_2 - \theta_2^d, \quad \tau_2 = -K_{\mathrm{P2}}(\theta_2 - \theta_2^d) - K_{\mathrm{I2}}\xi_2$$

PI control

$$\Downarrow$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} \cdots - K_{\mathrm{P1}}(\theta_1 - \theta_1^d) - K_{\mathrm{I1}}\xi_1 \\ \cdots - K_{\mathrm{P2}}(\theta_2 - \theta_2^d) - K_{\mathrm{I2}}\xi_2 \end{bmatrix}$$

$$\dot{\xi}_1 = \theta_1 - \theta_1^d$$

$$\dot{\xi}_2 = \theta_2 - \theta_2^d$$

current values of θ_1, θ_2, ω_1, ω_2, ξ_1, ξ_2

$$\Downarrow$$

their time derivatives $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\xi}_1$, $\dot{\xi}_2$

PD control (multiple desired values)

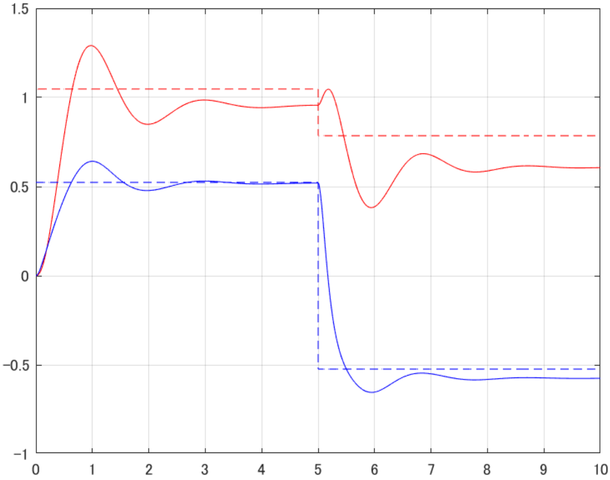
```
interval = [ 0, 5 ];
qinit = [0;0; 0;0];
thetad = [ pi/3; pi/6 ];
open_mechanism_2DOF_PD_ode = @(t,q) open_mechanism_2DOF_P
[time1, q1] = ode45(open_mechanism_2DOF_PD_ode, interval,

interval = [ 5, 10 ];
qinit = q1(end,:);
thetad = [ pi/4; -pi/6 ];
open_mechanism_2DOF_PD_ode = @(t,q) open_mechanism_2DOF_P
[time2, q2] = ode45(open_mechanism_2DOF_PD_ode, interval,

time = [time1;time2];
q = [q1;q2];
```

PD control (multiple desired values)

Result

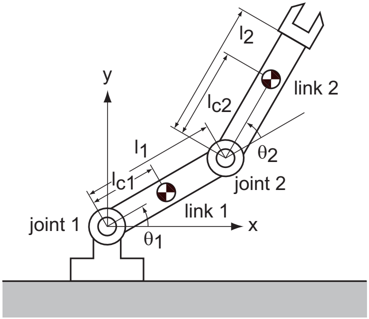


movie

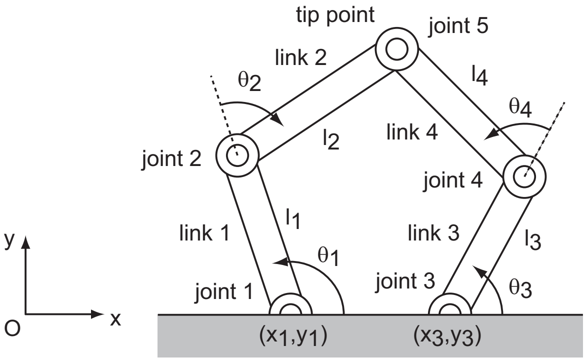
Report

Report #3 due date : Nov. 18 (Mon) 1:00 AM

Simulate the motion of a 2DOF open link mechanism under PID control. PID control is applied to active joints 1 and 2. Use appropriate values of geometrical and physical parameters of the manipulator.



Kinematics of 2DOF closed link mechanism



joint 1, 3: active
joint 2, 4, 5: passive

θ_1, θ_2, θ_3, θ_4:
rotation angles

τ_1, τ_3:
actuator torques

Kinematics of 2DOF closed link mechanism

decomposition of closed link mechanism into open link mechanisms:

left arm	link 1 and 2
right arm	link 3 and 4

end point of the left arm:

$$\boldsymbol{x}_{1,2} = \begin{bmatrix} x_{1,2} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_2 \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

end point of the right arm:

$$\boldsymbol{x}_{3,4} = \begin{bmatrix} x_{3,4} \\ y_{3,4} \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + l_3 \begin{bmatrix} C_3 \\ S_3 \end{bmatrix} + l_4 \begin{bmatrix} C_{3+4} \\ S_{3+4} \end{bmatrix}$$

Kinematics of 2DOF closed link mechanism

constraint vector:

$$\boldsymbol{R} \stackrel{\triangle}{=} \boldsymbol{x}_{1,2} - \boldsymbol{x}_{3,4} = \boldsymbol{0}$$

components of vector $\boldsymbol{R}$:

$$X \stackrel{\triangle}{=} x_{1,2} - x_{3,4} = l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3$$
$$Y \stackrel{\triangle}{=} y_{1,2} - y_{3,4} = l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3$$

Kinematics of 2DOF closed link mechanism

Jacobian of left arm:

$$J_{1,2} = \begin{bmatrix} \dfrac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1} & \dfrac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \partial x_{1,2}/\partial \theta_1 & \partial x_{1,2}/\partial \theta_2 \\ \partial y_{1,2}/\partial \theta_1 & \partial y_{1,2}/\partial \theta_2 \end{bmatrix}$$
$$= \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ l_1 C_1 + l_2 C_{1+2} & l_2 C_{1+2} \end{bmatrix}$$

Jacobian of right arm:

$$J_{3,4} = \begin{bmatrix} \dfrac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_3} & \dfrac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_4} \end{bmatrix} = \begin{bmatrix} \partial x_{3,4}/\partial \theta_3 & \partial x_{3,4}/\partial \theta_4 \\ \partial y_{3,4}/\partial \theta_3 & \partial y_{3,4}/\partial \theta_4 \end{bmatrix}$$
$$= \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ l_3 C_3 + l_4 C_{3+4} & l_4 C_{3+4} \end{bmatrix}$$

Lagrangian

Lagrangian of the closed link mechanism:

$$\mathcal{L} = \mathcal{L}_{1,2} + \mathcal{L}_{3,4} + \boldsymbol{\lambda}^{\top} \boldsymbol{R}$$

$\mathcal{L}_{1,2}$, $\mathcal{L}_{3,4}$	Lagrangians of the left and right arms
$\boldsymbol{\lambda} = [\, \lambda_x, \, \lambda_y \,]^{\top}$	Lagrange multiplier vector

Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}_{1,2}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}_{1,2}} = \mathbf{0}$$
$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}_{3,4}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}_{3,4}} = \mathbf{0}$$

where

$$\boldsymbol{\theta}_{1,2} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \; \boldsymbol{\omega}_{1,2} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \; \boldsymbol{\theta}_{3,4} = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \; \boldsymbol{\omega}_{3,4} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}$$

Contributions of $\mathcal{L}_{1,2}$

contributions of Lagrangian $\mathcal{L}_{1,2}$ to the Lagrange eqs:

$$- H_{1,2}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{\tau}_{1,2} + \boldsymbol{\tau}_{left}$$

$$\mathbf{0}$$

where

$$H_{1,2} = \begin{bmatrix} *** & J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) \\ J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) & J_2 + m_2 l_{c2}^2 \end{bmatrix}$$

$$\boldsymbol{\tau}_{1,2} = \begin{bmatrix} +h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 \\ -h_{12}\omega_1^2 + G_2 \end{bmatrix}$$

$$\boldsymbol{\tau}_{left} = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

$$*** = J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} C_2)$$

Contributions of $\mathcal{L}_{3,4}$

contributions of Lagrangian $\mathcal{L}_{3,4}$ to the Lagrange eqs:

$$\mathbf{0}$$

$$- H_{3,4}\,\dot{\boldsymbol{\omega}}_{3,4} + \boldsymbol{\tau}_{3,4} + \boldsymbol{\tau}_{right}$$

where

$$H_{3,4} = \begin{bmatrix} *** & J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) \\ J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) & J_4 + m_4 l_{c4}^2 \end{bmatrix}$$

$$\boldsymbol{\tau}_{3,4} = \begin{bmatrix} +h_{34}\omega_4^2 + 2h_{34}\omega_3\omega_4 + G_3 + G_4 \\ -h_{34}\omega_3^2 + G_4 \end{bmatrix}$$

$$\boldsymbol{\tau}_{right} = \begin{bmatrix} \tau_3 \\ 0 \end{bmatrix}$$

$$*** = J_3 + m_3 l_{c3}^2 + J_4 + m_4(l_3^2 + l_{c4}^2 + 2 l_3 l_{c4} C_4)$$

Contributions of $\boldsymbol{\lambda}^\top R$

since $\boldsymbol{x}_{3,4}$ is independent of θ_1 and θ_2

$$\frac{\partial \boldsymbol{R}}{\partial \theta_1} = \frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1}, \qquad \frac{\partial \boldsymbol{R}}{\partial \theta_2} = \frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2}$$

contributions of $\boldsymbol{\lambda}^\top \boldsymbol{R}$ to the first Lagrange eq:

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{\lambda}^\top \partial \boldsymbol{R}/\partial\theta_1 \\ \boldsymbol{\lambda}^\top \partial \boldsymbol{R}/\partial\theta_2 \end{bmatrix} &= \begin{bmatrix} \boldsymbol{\lambda}^\top \partial \boldsymbol{x}_{1,2}/\partial\theta_1 \\ \boldsymbol{\lambda}^\top \partial \boldsymbol{x}_{1,2}/\partial\theta_2 \end{bmatrix} = \begin{bmatrix} (\partial \boldsymbol{x}_{1,2}/\partial\theta_1)^\top \boldsymbol{\lambda} \\ (\partial \boldsymbol{x}_{1,2}/\partial\theta_2)^\top \boldsymbol{\lambda} \end{bmatrix} \\
&= \begin{bmatrix} (\partial \boldsymbol{x}_{1,2}/\partial\theta_1)^\top \\ (\partial \boldsymbol{x}_{1,2}/\partial\theta_2)^\top \end{bmatrix} \boldsymbol{\lambda} \\
&= \begin{bmatrix} \dfrac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1} & \dfrac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2} \end{bmatrix}^\top \boldsymbol{\lambda} \\
&= J_{1,2}^\top \boldsymbol{\lambda}
\end{aligned}$$

Contributions of $\boldsymbol{\lambda}^\top R$

since $\boldsymbol{x}_{1,2}$ is independent of θ_3 and θ_4

$$\frac{\partial \boldsymbol{R}}{\partial \theta_3} = -\frac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_3}, \qquad \frac{\partial \boldsymbol{R}}{\partial \theta_4} = -\frac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_4}$$

contributions of $\boldsymbol{\lambda}^\top \boldsymbol{R}$ to the second Lagrange eq:

$$\begin{aligned}
\left[\begin{array}{c} \boldsymbol{\lambda}^\top \partial \boldsymbol{R}/\partial \theta_3 \\ \boldsymbol{\lambda}^\top \partial \boldsymbol{R}/\partial \theta_4 \end{array}\right] &= \left[\begin{array}{c} -\boldsymbol{\lambda}^\top \partial \boldsymbol{x}_{3,4}/\partial \theta_3 \\ -\boldsymbol{\lambda}^\top \partial \boldsymbol{x}_{3,4}/\partial \theta_4 \end{array}\right] = \left[\begin{array}{c} -(\partial \boldsymbol{x}_{3,4}/\partial \theta_3)^\top \boldsymbol{\lambda} \\ -(\partial \boldsymbol{x}_{3,4}/\partial \theta_4)^\top \boldsymbol{\lambda} \end{array}\right] \\
&= \left[\begin{array}{c} -(\partial \boldsymbol{x}_{3,4}/\partial \theta_3)^\top \\ -(\partial \boldsymbol{x}_{3,4}/\partial \theta_4)^\top \end{array}\right] \boldsymbol{\lambda} \\
&= -\left[\begin{array}{cc} \dfrac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_3} & \dfrac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_4} \end{array}\right]^\top \boldsymbol{\lambda} \\
&= -J_{3,4}^\top \boldsymbol{\lambda}
\end{aligned}$$

Contributions of $\boldsymbol{\lambda}^\top R$

contributions of constraint term $\boldsymbol{\lambda}^\top \boldsymbol{R}$ to the Lagrange eqs:

$$J_{1,2}^\top \boldsymbol{\lambda}$$
$$-J_{3,4}^\top \boldsymbol{\lambda}$$

where $J_{1,2}$ and $J_{3,4}$ are Jacobians:

$$J_{1,2} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ l_1 C_1 + l_2 C_{1+2} & l_2 C_{1+2} \end{bmatrix}$$
$$J_{3,4} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ l_3 C_3 + l_4 C_{3+4} & l_4 C_{3+4} \end{bmatrix}$$

Lagrange equations of motion

$$-H_{1,2}\,\dot{\boldsymbol{\omega}}_{1,2} + \boldsymbol{\tau}_{1,2} + \boldsymbol{\tau}_{left} + J_{1,2}^{\top}\boldsymbol{\lambda} = \mathbf{0}$$
$$-H_{3,4}\,\dot{\boldsymbol{\omega}}_{3,4} + \boldsymbol{\tau}_{3,4} + \boldsymbol{\tau}_{right} - J_{3,4}^{\top}\boldsymbol{\lambda} = \mathbf{0}$$

$$\Downarrow$$

$$\begin{bmatrix} H_{1,2} & O_{2\times 2} & -J_{1,2}^{\top} \\ O_{2\times 2} & H_{3,4} & J_{3,4}^{\top} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{1,2} + \boldsymbol{\tau}_{left} \\ \boldsymbol{\tau}_{3,4} + \boldsymbol{\tau}_{right} \end{bmatrix}$$

Equation stabilizing constraint

constraint vector

$$\boldsymbol{R} = \boldsymbol{x}_{1,2}(\theta_1, \theta_2) - \boldsymbol{x}_{3,4}(\theta_3, \theta_4)$$

time-derivative

$$\begin{aligned} \dot{\boldsymbol{R}} &= \frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1}\omega_1 + \frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2}\omega_2 - \frac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_3}\omega_3 - \frac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_4}\omega_4 \\ &= \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1} & \frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2} \end{array} \right] \left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] - \left[\begin{array}{cc} \frac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_3} & \frac{\partial \boldsymbol{x}_{3,4}}{\partial \theta_4} \end{array} \right] \left[\begin{array}{c} \omega_3 \\ \omega_4 \end{array} \right] \\ &= J_{1,2}\boldsymbol{\omega}_{1,2} - J_{3,4}\boldsymbol{\omega}_{3,4} \end{aligned}$$

second-order time-derivative

$$\ddot{\boldsymbol{R}} = \dot{J}_{1,2}\boldsymbol{\omega}_{1,2} + J_{1,2}\dot{\boldsymbol{\omega}}_{1,2} - \dot{J}_{3,4}\boldsymbol{\omega}_{3,4} - J_{3,4}\dot{\boldsymbol{\omega}}_{3,4}$$

Equation stabilizing constraint

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1} = \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_1 \partial \theta_1}\omega_1 + \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_1 \partial \theta_2}\omega_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2} = \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_2 \partial \theta_1}\omega_1 + \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_2 \partial \theta_2}\omega_2$$

introduce Hessian matrices

$$Q_{1,2;x} = \begin{bmatrix} \dfrac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_1} & \dfrac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_2} \\ \dfrac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_1} & \dfrac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 C_1 - l_2 C_{1+2} & -l_2 C_{1+2} \\ -l_2 C_{1+2} & -l_2 C_{1+2} \end{bmatrix}$$

$$Q_{1,2;y} = \begin{bmatrix} \dfrac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_1} & \dfrac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_2} \\ \dfrac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_1} & \dfrac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ -l_2 S_{1+2} & -l_2 S_{1+2} \end{bmatrix}$$

Equation stabilizing constraint

$$
\begin{aligned}
j_{1,2}\boldsymbol{\omega}_{1,2} &= \left[\begin{array}{cc} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_1} & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{x}_{1,2}}{\partial \theta_2} \end{array}\right] \left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right] \\
&= \left[\begin{array}{cc} \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_1 \partial \theta_1}\omega_1 + \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_1 \partial \theta_2}\omega_2 & \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_2 \partial \theta_1}\omega_1 + \frac{\partial^2 \boldsymbol{x}_{1,2}}{\partial \theta_2 \partial \theta_2}\omega_2 \end{array}\right] \left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_1}\omega_1^2 + \frac{\partial^2 x_{1,2}}{\partial \theta_1 \partial \theta_2}\omega_1\omega_2 + \frac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_1}\omega_2\omega_1 + \frac{\partial^2 x_{1,2}}{\partial \theta_2 \partial \theta_2}\omega_2^2 \\ \frac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_1}\omega_1^2 + \frac{\partial^2 y_{1,2}}{\partial \theta_1 \partial \theta_2}\omega_1\omega_2 + \frac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_1}\omega_2\omega_1 + \frac{\partial^2 y_{1,2}}{\partial \theta_2 \partial \theta_2}\omega_2^2 \end{array}\right] \\
&= \left[\begin{array}{c} \left[\begin{array}{cc} \omega_1 & \omega_2 \end{array}\right] Q_{1,2;x} \left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right] \\ \left[\begin{array}{cc} \omega_1 & \omega_2 \end{array}\right] Q_{1,2;y} \left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right] \end{array}\right] = \left[\begin{array}{c} \boldsymbol{\omega}_{1,2}^{\top}\, Q_{1,2;x}\, \boldsymbol{\omega}_{1,2} \\ \boldsymbol{\omega}_{1,2}^{\top}\, Q_{1,2;y}\, \boldsymbol{\omega}_{1,2} \end{array}\right]
\end{aligned}
$$

Equation stabilizing constraint

similarly

$$j_{3,4}\boldsymbol{\omega}_{3,4} = \begin{bmatrix} \boldsymbol{\omega}_{3,4}^{\top}\, Q_{3,4;x}\, \boldsymbol{\omega}_{3,4} \\ \boldsymbol{\omega}_{3,4}^{\top}\, Q_{3,4;y}\, \boldsymbol{\omega}_{3,4} \end{bmatrix}$$

where Hessian matrices are

$$Q_{3,4;x} = \begin{bmatrix} \dfrac{\partial^2 x_{3,4}}{\partial\theta_3\partial\theta_3} & \dfrac{\partial^2 x_{3,4}}{\partial\theta_3\partial\theta_3} \\ \dfrac{\partial^2 x_{3,4}}{\partial\theta_4\partial\theta_4} & \dfrac{\partial^2 x_{3,4}}{\partial\theta_4\partial\theta_4} \end{bmatrix} = \begin{bmatrix} -l_3 C_3 - l_4 C_{3+4} & -l_4 C_{3+4} \\ -l_4 C_{3+4} & -l_4 C_{3+4} \end{bmatrix}$$

$$Q_{3,4;y} = \begin{bmatrix} \dfrac{\partial^2 y_{3,4}}{\partial\theta_3\partial\theta_3} & \dfrac{\partial^2 y_{3,4}}{\partial\theta_3\partial\theta_3} \\ \dfrac{\partial^2 y_{3,4}}{\partial\theta_4\partial\theta_4} & \dfrac{\partial^2 y_{3,4}}{\partial\theta_4\partial\theta_4} \end{bmatrix} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ -l_4 S_{3+4} & -l_4 S_{3+4} \end{bmatrix}$$

Equation stabilizing constraint

$$\ddot{\boldsymbol{R}} + 2\alpha\dot{\boldsymbol{R}} + \alpha^2\boldsymbol{R} = \boldsymbol{0}$$

$$\Downarrow$$

$$\begin{bmatrix} \boldsymbol{\omega}_{1,2}^\top Q_{1,2;x}\,\boldsymbol{\omega}_{1,2} \\ \boldsymbol{\omega}_{1,2}^\top Q_{1,2;y}\,\boldsymbol{\omega}_{1,2} \end{bmatrix} + J_{1,2}\dot{\boldsymbol{\omega}}_{1,2} - \begin{bmatrix} \boldsymbol{\omega}_{3,4}^\top Q_{3,4;x}\,\boldsymbol{\omega}_{3,4} \\ \boldsymbol{\omega}_{3,4}^\top Q_{3,4;y}\,\boldsymbol{\omega}_{3,4} \end{bmatrix} - J_{3,4}\dot{\boldsymbol{\omega}}_{3,4}$$
$$+2\alpha(J_{1,2}\boldsymbol{\omega}_{1,2} - J_{3,4}\boldsymbol{\omega}_{3,4}) + \alpha^2\boldsymbol{R} = \boldsymbol{0}$$

$$\Downarrow$$

Equation stabilizing constraint

$$\begin{bmatrix} -J_{1,2} & J_{3,4} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \end{bmatrix} = \boldsymbol{C}$$

where

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{\omega}_{1,2}^{\top} \, Q_{1,2;x} \, \boldsymbol{\omega}_{1,2} \\ \boldsymbol{\omega}_{1,2}^{\top} \, Q_{1,2;y} \, \boldsymbol{\omega}_{1,2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\omega}_{3,4}^{\top} \, Q_{3,4;x} \, \boldsymbol{\omega}_{3,4} \\ \boldsymbol{\omega}_{3,4}^{\top} \, Q_{3,4;y} \, \boldsymbol{\omega}_{3,4} \end{bmatrix} + 2\alpha(J_{1,2}\boldsymbol{\omega}_{1,2} - J_{3,4}\boldsymbol{\omega}_{3,4}) + \alpha^2 \boldsymbol{R}$$

Dynamic equations for closed link mechanism

Combining Lagrange equation of motion and equation stabilizing constraint yields

$$\begin{bmatrix} H_{1,2} & O_{2\times 2} & -J_{1,2}^{\top} \\ O_{2\times 2} & H_{3,4} & J_{3,4}^{\top} \\ -J_{1,2} & J_{3,4} & O_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{1,2} + \boldsymbol{\tau}_{left} \\ \boldsymbol{\tau}_{3,4} + \boldsymbol{\tau}_{right} \\ \boldsymbol{C} \end{bmatrix}$$

coefficient matrix is regular $\longrightarrow$ we can compute $\dot{\omega}_1$ through $\dot{\omega}_4$

Physical Interpretation

$J_{1,2}$ and $J_{3,4}$: Jacobian matrices of the left and right arms
$\boldsymbol{\lambda} = [\,\lambda_x,\ \lambda_y\,]^\top$: constraint force
equivalent torques around rotational joints 1 and 2:

$$J_{1,2}^\top \boldsymbol{\lambda} = \begin{bmatrix} \lambda_x(-l_1 S_1 - l_2 S_{1+2}) + \lambda_y(l_1 C_1 + l_2 C_{1+2}) \\ \lambda_x(-l_2 S_{1+2}) + \lambda_y l_2 C_{1+2} \end{bmatrix}$$

reaction force $-\boldsymbol{\lambda}$
equivalent torques around rotational joint 3 and 4:

$$J_{3,4}^\top(-\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_x(l_3 S_3 + l_4 S_{3+4}) + \lambda_y(-l_3 C_3 - l_4 C_{3+4}) \\ \lambda_x l_4 S_{3+4} + \lambda_y(-l_4 C_{3+4}) \end{bmatrix}$$

PD control

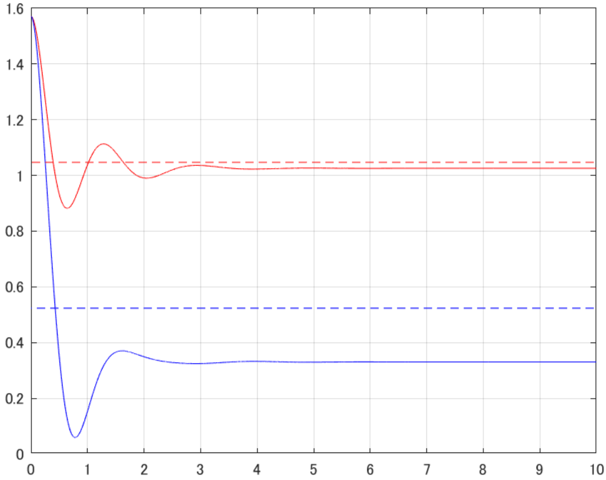
$$\tau_1 = -K_{\mathrm{P1}}(\theta_1 - \theta_1^d) - K_{\mathrm{D1}}\dot{\theta}_1$$
$$\tau_3 = -K_{\mathrm{P3}}(\theta_3 - \theta_3^d) - K_{\mathrm{D3}}\dot{\theta}_3$$

Sample Programs

- class **Closed_Mechanism_Two_DOF**
- closed_mechanism_2DOF_PD.m
  PD control of 2DOF closed mechanism
- closed_mechanism_2DOF_PD_params.m equation of motion

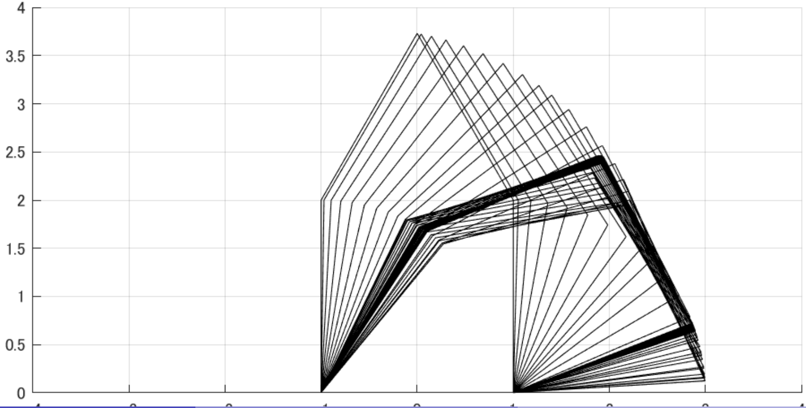
PD control

Result



PD control

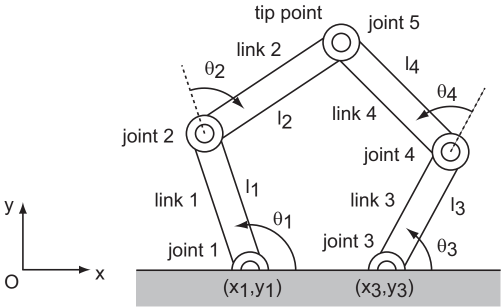
Result



Report

Report #4   due date : Nov. 25 (Mon) 1:00 AM

Simulate the motion of a 2DOF closed link mechanism under PID control. PID control is applied to active joints 1 and 3. Use appropriate values of geometrical and physical parameters of the manipulator.



Summary

Open link mechanism

- inertia matrix depends on joint angles
- Lagrange equations of motion of open link mechanism

Closed link mechanism

- two open link mechanisms with geometric constraints
- synthesized from Lagrange equations of open link mechanisms