

Analytical Mechanics: Link Mechanisms

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Kinetic energy

velocity of the center of mass of link 1:

$$\dot{\mathbf{x}}_{c1} = l_{c1}\dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}$$

angular velocity of link 1:

$$\dot{\theta}_1$$

kinetic energy of link 1:

$$\begin{aligned} T_1 &= \frac{1}{2}m_1\dot{\mathbf{x}}_{c1}^T\dot{\mathbf{x}}_{c1} + \frac{1}{2}J_1\dot{\theta}_1^2 \\ &= \frac{1}{2}(m_1l_{c1}^2 + J_1)\dot{\theta}_1^2 \end{aligned}$$

Kinetic energy

velocity of the center of mass of link 2:

$$\dot{\mathbf{x}}_{c2} = l_1\dot{\theta}_1 \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix} + l_{c2}(\dot{\theta}_1 + \dot{\theta}_2) \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

angular velocity of link 2:

$$\dot{\theta}_1 + \dot{\theta}_2$$

kinetic energy of link 2:

$$\begin{aligned} T_2 &= \frac{1}{2}m_2\dot{\mathbf{x}}_{c2}^T\dot{\mathbf{x}}_{c2} + \frac{1}{2}J_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &= \frac{1}{2}m_2\{l_1^2\dot{\theta}_1^2 + l_{c2}^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2l_1l_{c2}C_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\} + \\ &\quad \frac{1}{2}J_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned}$$

Kinetic energy

total kinetic energy

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

where

$$\begin{aligned} H_{11} &= J_1 + m_1l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2l_1l_{c2}C_2) \\ H_{22} &= J_2 + m_2l_{c2}^2 \\ H_{12} &= H_{21} = J_2 + m_2(l_{c2}^2 + l_1l_{c2}C_2) \end{aligned}$$

inertia matrix

$$H \triangleq \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Partial derivatives

H_{11} and $H_{12} = H_{21}$ depend on θ_2 :

$$\frac{\partial H_{11}}{\partial \theta_2} = -2h_{12}, \quad \frac{\partial H_{12}}{\partial \theta_2} = \frac{\partial H_{21}}{\partial \theta_2} = -h_{12} \quad (h_{12} \triangleq m_2l_1l_{c2}S_2)$$

$$\dot{H}_{11} = -2h_{12}\dot{\theta}_2, \quad \dot{H}_{12} = \dot{H}_{21} = -h_{12}\dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = H_{11}\dot{\theta}_1 + H_{12}\dot{\theta}_2, \quad \frac{\partial T}{\partial \dot{\theta}_2} = H_{21}\dot{\theta}_1 + H_{22}\dot{\theta}_2$$

$$-\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_1} = -\dot{H}_{11}\dot{\theta}_1 - H_{11}\ddot{\theta}_1 - \dot{H}_{12}\dot{\theta}_2 - H_{12}\ddot{\theta}_2$$

$$= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2$$

$$-\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_2} = -\dot{H}_{21}\dot{\theta}_1 - H_{21}\ddot{\theta}_1 - \dot{H}_{22}\dot{\theta}_2 - H_{22}\ddot{\theta}_2$$

$$= h_{12}\dot{\theta}_1\dot{\theta}_2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2$$

Agenda

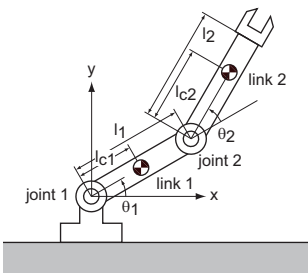
1 Open Link Mechanism

- Kinematics of Open Link Mechanism
- Dynamics of 2DOF open link mechanism

2 Closed Link Mechanism

- Kinematics of Closed Link Mechanism
- Dynamics of 2DOF closed link mechanism

Kinematics of 2DOF open link mechanism



two link open link mechanism

- l_i length of link i
- l_{ci} distance btw. joint i and the center of mass of link i
- m_i mass of link i
- J_i inertia of moment of link i around its center of mass
- θ_1 rotation angle of joint 1
- θ_2 rotation angle of joint 2

Kinematics of 2DOF open link mechanism

position of the center of mass of link 1:

$$\mathbf{x}_{c1} \triangleq \begin{bmatrix} x_{c1} \\ y_{c1} \end{bmatrix} = l_{c1} \begin{bmatrix} C_1 \\ S_1 \end{bmatrix}$$

position of the center of mass of link 2:

$$\mathbf{x}_{c2} \triangleq \begin{bmatrix} x_{c2} \\ y_{c2} \end{bmatrix} = l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_{c2} \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

orientation angle of link 1:

$$\theta_1$$

orientation angle of link 2:

$$\theta_1 + \theta_2$$

Partial derivatives

H_{11} , H_{22} , and $H_{12} = H_{21}$ are independent of θ_1

$$\frac{\partial T}{\partial \theta_1} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = 0$$

H_{11} and $H_{12} = H_{21}$ depend on θ_2

$$\frac{\partial T}{\partial \theta_2} = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -2h_{12} & -h_{12} \\ -h_{12} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = -h_{12}\dot{\theta}_1^2 - h_{12}\dot{\theta}_1\dot{\theta}_2$$

contribution of kinetic energy:

$$\begin{aligned} \frac{\partial T}{\partial \theta_1} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_1} &= 2h_{12}\dot{\theta}_1\dot{\theta}_2 + h_{12}\dot{\theta}_2^2 - H_{11}\ddot{\theta}_1 - H_{12}\ddot{\theta}_2 \\ \frac{\partial T}{\partial \theta_2} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_2} &= -h_{12}\dot{\theta}_1^2 - H_{21}\ddot{\theta}_1 - H_{22}\ddot{\theta}_2 \end{aligned}$$

Lagrange equations of motion

canonical form of ordinary differential equations:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 + \tau_1 \\ -h_{12}\omega_1^2 + G_2 + \tau_2 \end{bmatrix}$$

state variables: joint angles θ_1 , θ_2 and angular velocities ω_1 , ω_2

the inertia matrix is regular $\rightarrow$ 2nd eq. is solvable

$\rightarrow$ we can compute $\dot{\omega}_1$ and $\dot{\omega}_2$

$\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\omega}_1$, $\dot{\omega}_2$ are functions of θ_1 , θ_2 , ω_1 , ω_2

$\Downarrow$

we can sketch θ_1 , θ_2 , ω_1 , ω_2 using an ODE solver.

Gravitational potential energy

gravitational acceleration vector:

$$\mathbf{g} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

potential energies of link 1 and 2:

$$U_1 = -m_1 \mathbf{g}^\top \mathbf{x}_{c1}, \quad U_2 = -m_2 \mathbf{g}^\top \mathbf{x}_{c2}$$

potential energy:

$$U = U_1 + U_2$$

contribution of potential energy:

$$-\frac{\partial U}{\partial \theta_1} = G_1 + G_2, \quad -\frac{\partial U}{\partial \theta_2} = G_2$$

where

$$G_1 = (m_1 l_{c1} + m_2 l_1) \mathbf{g}^\top \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix}, \quad G_2 = m_2 l_{c2} \mathbf{g}^\top \begin{bmatrix} -S_{1+2} \\ C_{1+2} \end{bmatrix}$$

Work done by actuator torques

work done by τ_1 applied to rotational joint 1:

$$\tau_1 \theta_1$$

work done by τ_2 applied to rotational joint 2:

$$\tau_2 \theta_2$$

work done by the two actuator torques:

$$W = \tau_1 \theta_1 + \tau_2 \theta_2$$

contribution of work:

$$\frac{\partial W}{\partial \theta_1} = \tau_1, \quad \frac{\partial W}{\partial \theta_2} = \tau_2$$

Lagrange equations of motion

Lagrangian:

$$\mathcal{L} = T - U + W$$

Lagrange equations of motion

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta_1} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} &= 0 \\ \frac{\partial \mathcal{L}}{\partial \theta_2} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} &= 0 \end{aligned}$$

let $\omega_1 \triangleq \dot{\theta}_1$ and $\omega_2 \triangleq \dot{\theta}_2$:

$$\begin{aligned} -H_{11}\dot{\omega}_1 - H_{12}\dot{\omega}_2 + h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 + \tau_1 &= 0 \\ -H_{22}\dot{\omega}_2 - H_{12}\dot{\omega}_1 - h_{12}\omega_1^2 + G_2 + \tau_2 &= 0 \end{aligned}$$

Sample Programs

- class **Link**
- class **Link_Cylinder**
- class **Open_Mechanism_Two_DOF**
- class **Closed_Mechanism_Two_DOF**

class **Link_Cylinder** is a subclass of class **Link**

Sample Programs

file **Link.m**

```
classdef Link
    properties
        length;
        length_center;
        mass;
        inertia_of_moment_center;
        inertia_of_moment;
    end
    methods
        function obj = Link(l, lc, m, Jc, J)
            obj.length = l;
            obj.length_center = lc;
            obj.mass = m;
            obj.inertia_of_moment_center = Jc;
            obj.inertia_of_moment = J;
        end
    end
end
```

Sample Programs

Sentence

```
>> link1 = Link(2, 1, 0.0157, 0.0052, 0.0210)
```

builds a link with $l = 2$, $l_c = 1$, $m = 0.0157$, $J_c = 0.0052$, and $J = 0.0210$.

```
>> link1
link1 =
    Link properties:
        length: 2
    length_center: 1
        mass: 0.0157
    inertia_of_moment_center: 0.0052
    inertia_of_moment: 0.0210
>>
```

Sample Programs

building two cylindrical links of length 2, radius 0.05, and density 1

```
len = 2.00; radius = 0.05; density = 1;
```

```
len_c = len/2;
m = density * len * (pi*(radius)^2);
Jc = (1/12) * m * (3*radius^2 + len^2);
J = Jc + m * (len - len_c)^2;
```

```
link1 = Link (len, len_c, m, Jc, J);
link2 = Link (len, len_c, m, Jc, J);
```

```
>> link1
link1 =
  Link properties:
```

```
    length: 2
length_center: 1
    mass: 0.0157
```

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Sample Programs

calculating inertia matrix and torque vector

```
[ mat, vec ] = robot.inertia_matrix_and_torque_vector
```

$$\text{mat} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \text{vec} = \begin{bmatrix} h_{12}\omega_2^2 + 2h_{12}\omega_1\omega_2 + G_1 + G_2 \\ -h_{12}\omega_1^2 + G_2 \end{bmatrix}$$

Note vec does not include τ_1 or τ_2 .

Solving

$$\text{mat } \dot{\omega} = \text{vec} + \tau$$

where $\tau = [\tau_1, \tau_2]^T$, yields angular acceleration $\dot{\omega}$.

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Sample Programs

building two cylindrical links of length 2, radius 0.05, and density 1

```
len = 2.00; radius = 0.05; density = 1;
```

```
link1 = Link_Cylinder (len, radius, density);
link2 = Link_Cylinder (len, radius, density);
```

```
>> link1
link1 =
  Link_Cylinder properties:
```

```
    radius: 0.0500
    density: 1
    length: 2
length_center: 1
    mass: 0.0157
inertia_of_moment_center: 0.0052
inertia_of_moment: 0.0210
```

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Driving by external torques

Sample Programs

- [open_mechanism_2DOF_external_torques.m](#)
2DOF open mechanism driven by external torques
- [open_mechanism_2DOF_external_torques_params.m](#)
equation of motion

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Sample Programs

building an open mechanism consisting of two links

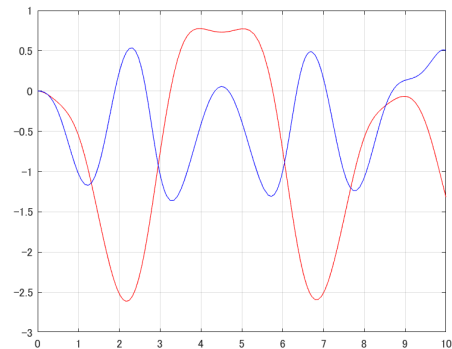
```
base = [0; 0];
grav = [0; -9.8];
robot = Open_Mechanism_Two_DOF (link1, link2, base, grav)
```

```
>> robot
robot =
  Open_Mechanism_Two_DOF properties:
    link1: [1 x 1 Link_Cylinder]
    link2: [1 x 1 Link_Cylinder]
base_position: [2 x 1 double]
gravity: [2 x 1 double]
theta1: []
theta2: []
omega1: []
omega2: []
C1: []
```

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Driving by external torques

Result



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Sample Programs

setting joint angles and angular velocities

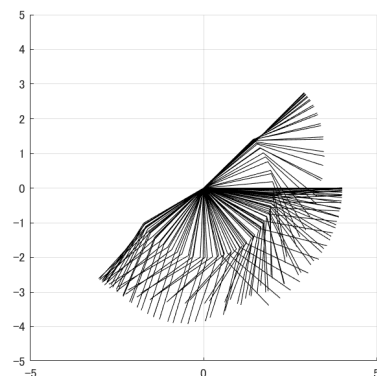
```
theta = [ pi/3; pi/6 ];
omega = [ 0; 0 ];
robot = robot.joint_angles (theta, omega);
```

```
>> robot
robot =
  Open_Mechanism_Two_DOF properties:
    link1: [1 x 1 Link_Cylinder]
    link2: [1 x 1 Link_Cylinder]
base_position: [2 x 1 double]
gravity: [2 x 1 double]
theta1: 1.0472
theta2: 0.5236
omega1: 0
omega2: 0
C1: 0.5000
```

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Driving by external torques

Result



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PD control

$$\begin{aligned}\tau_1 &= -K_{P1}(\theta_1 - \theta_1^d) - K_{D1}\dot{\theta}_1 \\ \tau_2 &= -K_{P2}(\theta_2 - \theta_2^d) - K_{D2}\dot{\theta}_2\end{aligned}$$

↓

$$\begin{aligned}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} &= \begin{bmatrix} \cdots - K_{P1}(\theta_1 - \theta_1^d) - K_{D1}\omega_1 \\ \cdots - K_{P2}(\theta_2 - \theta_2^d) - K_{D2}\omega_2 \end{bmatrix}\end{aligned}$$

current values of $\theta_1, \theta_2, \omega_1, \omega_2$

↓

their time derivatives $\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2$

PI control

$$\begin{aligned}\tau_1 &= -K_{P1}(\theta_1 - \theta_1^d) - K_{I1} \int_0^t \{(\theta_1 - \theta_1^d)(\tau)\} d\tau \\ \tau_2 &= -K_{P2}(\theta_2 - \theta_2^d) - K_{I2} \int_0^t \{(\theta_2 - \theta_2^d)(\tau)\} d\tau\end{aligned}$$

Introduce additional variables:

$$\xi_1 \triangleq \int_0^t \{(\theta_1 - \theta_1^d)(\tau)\} d\tau$$

$$\xi_2 \triangleq \int_0^t \{(\theta_2 - \theta_2^d)(\tau)\} d\tau$$

$$\begin{aligned}\dot{\xi}_1 &= \theta_1 - \theta_1^d, & \tau_1 &= -K_{P1}(\theta_1 - \theta_1^d) - K_{I1}\xi_1 \\ \dot{\xi}_2 &= \theta_2 - \theta_2^d, & \tau_2 &= -K_{P2}(\theta_2 - \theta_2^d) - K_{I2}\xi_2\end{aligned}$$

PD control

Sample Programs

- [open_mechanism_2DOF_PD.m](#)
PD control of 2DOF open mechanism
- [open_mechanism_2DOF_PD_params.m](#)
equation of motion

PI control

↓

$$\begin{aligned}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} &= \begin{bmatrix} \cdots - K_{P1}(\theta_1 - \theta_1^d) - K_{I1}\xi_1 \\ \cdots - K_{P2}(\theta_2 - \theta_2^d) - K_{I2}\xi_2 \end{bmatrix} \\ \dot{\xi}_1 &= \theta_1 - \theta_1^d \\ \dot{\xi}_2 &= \theta_2 - \theta_2^d\end{aligned}$$

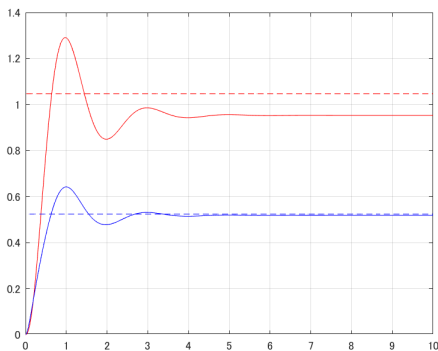
current values of $\theta_1, \theta_2, \omega_1, \omega_2, \xi_1, \xi_2$

↓

their time derivatives $\dot{\theta}_1, \dot{\theta}_2, \dot{\omega}_1, \dot{\omega}_2, \dot{\xi}_1, \dot{\xi}_2$

PD control

Result



PD control (multiple desired values)

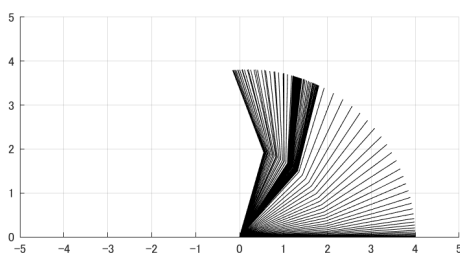
```
interval = [ 0, 5 ];
qinit = [0;0; 0;0];
thetad = [ pi/3; pi/6 ];
open_mechanism_2DOF_PD_ode = @(t,q) open_mechanism_2DOF_PD_ode;
[time1, q1] = ode45(open_mechanism_2DOF_PD_ode, interval, qinit);

interval = [ 5, 10 ];
qinit = q1(end,:);
thetad = [ pi/4; -pi/6 ];
open_mechanism_2DOF_PD_ode = @(t,q) open_mechanism_2DOF_PD_ode;
[time2, q2] = ode45(open_mechanism_2DOF_PD_ode, interval, qinit);

time = [time1;time2];
q = [q1;q2];
```

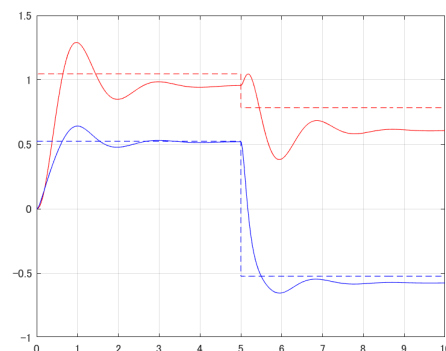
PD control

Result



PD control (multiple desired values)

Result

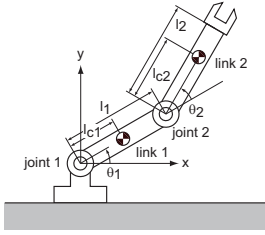


movie

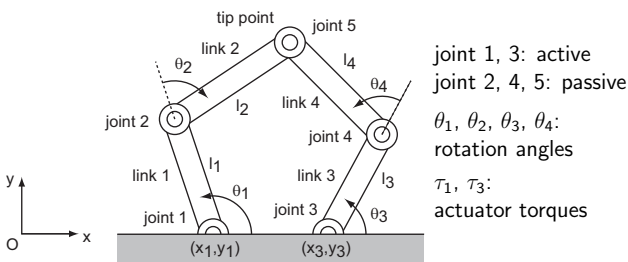
Report

Report #3 due date : Nov. 18 (Mon) 1:00 AM

Simulate the motion of a 2DOF open link mechanism under PID control. PID control is applied to active joints 1 and 2. Use appropriate values of geometrical and physical parameters of the manipulator.



Kinematics of 2DOF closed link mechanism



Kinematics of 2DOF closed link mechanism

decomposition of closed link mechanism into open link mechanisms:

left arm link 1 and 2

right arm link 3 and 4

end point of the left arm:

$$\mathbf{x}_{1,2} = \begin{bmatrix} x_{1,2} \\ y_{1,2} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + l_1 \begin{bmatrix} C_1 \\ S_1 \end{bmatrix} + l_2 \begin{bmatrix} C_{1+2} \\ S_{1+2} \end{bmatrix}$$

end point of the right arm:

$$\mathbf{x}_{3,4} = \begin{bmatrix} x_{3,4} \\ y_{3,4} \end{bmatrix} = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + l_3 \begin{bmatrix} C_3 \\ S_3 \end{bmatrix} + l_4 \begin{bmatrix} C_{3+4} \\ S_{3+4} \end{bmatrix}$$

Kinematics of 2DOF closed link mechanism

constraint vector:

$$\mathbf{R} \triangleq \mathbf{x}_{1,2} - \mathbf{x}_{3,4} = \mathbf{0}$$

components of vector $\mathbf{R}$:

$$X \triangleq x_{1,2} - x_{3,4} = l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3$$

$$Y \triangleq y_{1,2} - y_{3,4} = l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3$$

Kinematics of 2DOF closed link mechanism

Jacobian of left arm:

$$\begin{aligned} J_{1,2} &= \begin{bmatrix} \frac{\partial x_{1,2}}{\partial \theta_1} & \frac{\partial x_{1,2}}{\partial \theta_2} \\ \frac{\partial y_{1,2}}{\partial \theta_1} & \frac{\partial y_{1,2}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{1,2}}{\partial \theta_1} & \frac{\partial x_{1,2}}{\partial \theta_2} \\ \frac{\partial y_{1,2}}{\partial \theta_1} & \frac{\partial y_{1,2}}{\partial \theta_2} \end{bmatrix} \\ &= \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ l_1 C_1 + l_2 C_{1+2} & l_2 C_{1+2} \end{bmatrix} \end{aligned}$$

Jacobian of right arm:

$$\begin{aligned} J_{3,4} &= \begin{bmatrix} \frac{\partial x_{3,4}}{\partial \theta_3} & \frac{\partial x_{3,4}}{\partial \theta_4} \\ \frac{\partial y_{3,4}}{\partial \theta_3} & \frac{\partial y_{3,4}}{\partial \theta_4} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{3,4}}{\partial \theta_3} & \frac{\partial x_{3,4}}{\partial \theta_4} \\ \frac{\partial y_{3,4}}{\partial \theta_3} & \frac{\partial y_{3,4}}{\partial \theta_4} \end{bmatrix} \\ &= \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ l_3 C_3 + l_4 C_{3+4} & l_4 C_{3+4} \end{bmatrix} \end{aligned}$$

Lagrangian

Lagrangian of the closed link mechanism:

$$\mathcal{L} = \mathcal{L}_{1,2} + \mathcal{L}_{3,4} + \lambda^\top \mathbf{R}$$

$\mathcal{L}_{1,2}, \mathcal{L}_{3,4}$ Lagrangians of the left and right arms

$\lambda = [\lambda_x, \lambda_y]^\top$ Lagrange multiplier vector

Lagrange equations of motion:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta_{1,2}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \omega_{1,2}} &= \mathbf{0} \\ \frac{\partial \mathcal{L}}{\partial \theta_{3,4}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \omega_{3,4}} &= \mathbf{0} \end{aligned}$$

where

$$\theta_{1,2} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \omega_{1,2} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \theta_{3,4} = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix}, \omega_{3,4} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}$$

Contributions of $\mathcal{L}_{1,2}$

contributions of Lagrangian $\mathcal{L}_{1,2}$ to the Lagrange eqs:

$$\begin{aligned} &-H_{1,2} \dot{\omega}_{1,2} + \tau_{1,2} + \tau_{left} \\ &\mathbf{0} \end{aligned}$$

where

$$\begin{aligned} H_{1,2} &= \begin{bmatrix} *** & J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) \\ J_2 + m_2(l_{c2}^2 + l_1 l_{c2} C_2) & J_2 + m_2 l_{c2}^2 \end{bmatrix} \\ \tau_{1,2} &= \begin{bmatrix} +h_{12} \omega_2^2 + 2h_{12} \omega_1 \omega_2 + G_1 + G_2 \\ -h_{12} \omega_1^2 + G_2 \end{bmatrix} \\ \tau_{left} &= \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \\ *** &= J_1 + m_1 l_{c1}^2 + J_2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2} C_2) \end{aligned}$$

Contributions of $\mathcal{L}_{3,4}$

contributions of Lagrangian $\mathcal{L}_{3,4}$ to the Lagrange eqs:

$$\begin{aligned} &\mathbf{0} \\ &-H_{3,4} \dot{\omega}_{3,4} + \tau_{3,4} + \tau_{right} \end{aligned}$$

where

$$\begin{aligned} H_{3,4} &= \begin{bmatrix} *** & J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) \\ J_4 + m_4(l_{c4}^2 + l_3 l_{c4} C_4) & J_4 + m_4 l_{c4}^2 \end{bmatrix} \\ \tau_{3,4} &= \begin{bmatrix} +h_{34} \omega_4^2 + 2h_{34} \omega_3 \omega_4 + G_3 + G_4 \\ -h_{34} \omega_3^2 + G_4 \end{bmatrix} \\ \tau_{right} &= \begin{bmatrix} \tau_3 \\ 0 \end{bmatrix} \\ *** &= J_3 + m_3 l_{c3}^2 + J_4 + m_4(l_3^2 + l_{c4}^2 + 2l_3 l_{c4} C_4) \end{aligned}$$

Contributions of $\lambda^\top R$

since $\mathbf{x}_{3,4}$ is independent of θ_1 and θ_2

$$\frac{\partial R}{\partial \theta_1} = \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1}, \quad \frac{\partial R}{\partial \theta_2} = \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2}$$

contributions of $\lambda^\top R$ to the first Lagrange eq:

$$\begin{aligned} \begin{bmatrix} \lambda^\top \partial R / \partial \theta_1 \\ \lambda^\top \partial R / \partial \theta_2 \end{bmatrix} &= \begin{bmatrix} \lambda^\top \partial \mathbf{x}_{1,2} / \partial \theta_1 \\ \lambda^\top \partial \mathbf{x}_{1,2} / \partial \theta_2 \end{bmatrix} = \begin{bmatrix} (\partial \mathbf{x}_{1,2} / \partial \theta_1)^\top \lambda \\ (\partial \mathbf{x}_{1,2} / \partial \theta_2)^\top \lambda \end{bmatrix} \\ &= \begin{bmatrix} (\partial \mathbf{x}_{1,2} / \partial \theta_1)^\top \\ (\partial \mathbf{x}_{1,2} / \partial \theta_2)^\top \end{bmatrix} \lambda \\ &= \begin{bmatrix} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix}^\top \lambda \\ &= J_{1,2}^\top \lambda \end{aligned}$$

Equation stabilizing constraint

constraint vector

$$R = \mathbf{x}_{1,2}(\theta_1, \theta_2) - \mathbf{x}_{3,4}(\theta_3, \theta_4)$$

time-derivative

$$\begin{aligned} \dot{R} &= \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} \omega_1 + \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \omega_2 - \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} \omega_3 - \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \omega_4 \\ &= \begin{bmatrix} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} - \begin{bmatrix} \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} & \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \end{bmatrix} \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix} \\ &= J_{1,2} \omega_{1,2} - J_{3,4} \omega_{3,4} \end{aligned}$$

second-order time-derivative

$$\ddot{R} = \dot{J}_{1,2} \omega_{1,2} + J_{1,2} \dot{\omega}_{1,2} - \dot{J}_{3,4} \omega_{3,4} - J_{3,4} \dot{\omega}_{3,4}$$

Contributions of $\lambda^\top R$

since $\mathbf{x}_{1,2}$ is independent of θ_3 and θ_4

$$\frac{\partial R}{\partial \theta_3} = -\frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3}, \quad \frac{\partial R}{\partial \theta_4} = -\frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4}$$

contributions of $\lambda^\top R$ to the second Lagrange eq:

$$\begin{aligned} \begin{bmatrix} \lambda^\top \partial R / \partial \theta_3 \\ \lambda^\top \partial R / \partial \theta_4 \end{bmatrix} &= \begin{bmatrix} -\lambda^\top \partial \mathbf{x}_{3,4} / \partial \theta_3 \\ -\lambda^\top \partial \mathbf{x}_{3,4} / \partial \theta_4 \end{bmatrix} = \begin{bmatrix} -(\partial \mathbf{x}_{3,4} / \partial \theta_3)^\top \lambda \\ -(\partial \mathbf{x}_{3,4} / \partial \theta_4)^\top \lambda \end{bmatrix} \\ &= \begin{bmatrix} -(\partial \mathbf{x}_{3,4} / \partial \theta_3)^\top \\ -(\partial \mathbf{x}_{3,4} / \partial \theta_4)^\top \end{bmatrix} \lambda \\ &= -\begin{bmatrix} \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_3} & \frac{\partial \mathbf{x}_{3,4}}{\partial \theta_4} \end{bmatrix}^\top \lambda \\ &= -J_{3,4}^\top \lambda \end{aligned}$$

Contributions of $\lambda^\top R$

contributions of constraint term $\lambda^\top R$ to the Lagrange eqs:

$$\begin{aligned} &J_{1,2}^\top \lambda \\ &-J_{3,4}^\top \lambda \end{aligned}$$

where $J_{1,2}$ and $J_{3,4}$ are Jacobians:

$$\begin{aligned} J_{1,2} &= \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ l_1 C_1 + l_2 C_{1+2} & l_2 C_{1+2} \end{bmatrix} \\ J_{3,4} &= \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ l_3 C_3 + l_4 C_{3+4} & l_4 C_{3+4} \end{bmatrix} \end{aligned}$$

Equation stabilizing constraint

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} &= \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_2 \\ \frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} &= \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2 \end{aligned}$$

introduce Hessian matrices

$$\begin{aligned} Q_{1,2;x} &= \begin{bmatrix} \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 C_1 - l_2 C_{1+2} & -l_2 C_{1+2} \\ -l_2 C_{1+2} & -l_2 C_{1+2} \end{bmatrix} \\ Q_{1,2;y} &= \begin{bmatrix} \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_1 \partial \theta_1} & \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_2 \partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 S_1 - l_2 S_{1+2} & -l_2 S_{1+2} \\ -l_2 S_{1+2} & -l_2 S_{1+2} \end{bmatrix} \end{aligned}$$

Equation stabilizing constraint

Equation stabilizing constraint

$$\begin{aligned} j_{1,2} \omega_{1,2} &= \begin{bmatrix} \frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_1} & \frac{d}{dt} \frac{\partial \mathbf{x}_{1,2}}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_2 & \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1^2 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_1 \omega_2 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_2 \omega_1 + \frac{\partial^2 \mathbf{x}_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2^2 \\ \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_1 \partial \theta_1} \omega_1^2 + \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_1 \partial \theta_2} \omega_1 \omega_2 + \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_2 \partial \theta_1} \omega_2 \omega_1 + \frac{\partial^2 \mathbf{y}_{1,2}}{\partial \theta_2 \partial \theta_2} \omega_2^2 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} Q_{1,2;x} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} Q_{1,2;y} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \omega_{1,2}^\top Q_{1,2;x} \omega_{1,2} \\ \omega_{1,2}^\top Q_{1,2;y} \omega_{1,2} \end{bmatrix} \end{aligned}$$

Lagrange equations of motion

$$\begin{aligned} -H_{1,2} \dot{\omega}_{1,2} + \tau_{1,2} + \tau_{left} + J_{1,2}^\top \lambda &= 0 \\ -H_{3,4} \dot{\omega}_{3,4} + \tau_{3,4} + \tau_{right} - J_{3,4}^\top \lambda &= 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{bmatrix} H_{1,2} & O_{2 \times 2} & -J_{1,2}^\top \\ O_{2 \times 2} & H_{3,4} & J_{3,4}^\top \end{bmatrix} \begin{bmatrix} \dot{\omega}_{1,2} \\ \dot{\omega}_{3,4} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tau_{1,2} + \tau_{left} \\ \tau_{3,4} + \tau_{right} \\ 0 \end{bmatrix}$$

Equation stabilizing constraint

similarly

$$j_{3,4} \omega_{3,4} = \begin{bmatrix} \omega_{3,4}^\top Q_{3,4;x} \omega_{3,4} \\ \omega_{3,4}^\top Q_{3,4;y} \omega_{3,4} \end{bmatrix}$$

where Hessian matrices are

$$\begin{aligned} Q_{3,4;x} &= \begin{bmatrix} \frac{\partial^2 \mathbf{x}_{3,4}}{\partial \theta_3 \partial \theta_3} & \frac{\partial^2 \mathbf{x}_{3,4}}{\partial \theta_3 \partial \theta_4} \\ \frac{\partial^2 \mathbf{x}_{3,4}}{\partial \theta_4 \partial \theta_3} & \frac{\partial^2 \mathbf{x}_{3,4}}{\partial \theta_4 \partial \theta_4} \end{bmatrix} = \begin{bmatrix} -l_3 C_3 - l_4 C_{3+4} & -l_4 C_{3+4} \\ -l_4 C_{3+4} & -l_4 C_{3+4} \end{bmatrix} \\ Q_{3,4;y} &= \begin{bmatrix} \frac{\partial^2 \mathbf{y}_{3,4}}{\partial \theta_3 \partial \theta_3} & \frac{\partial^2 \mathbf{y}_{3,4}}{\partial \theta_3 \partial \theta_4} \\ \frac{\partial^2 \mathbf{y}_{3,4}}{\partial \theta_4 \partial \theta_3} & \frac{\partial^2 \mathbf{y}_{3,4}}{\partial \theta_4 \partial \theta_4} \end{bmatrix} = \begin{bmatrix} -l_3 S_3 - l_4 S_{3+4} & -l_4 S_{3+4} \\ -l_4 S_{3+4} & -l_4 S_{3+4} \end{bmatrix} \end{aligned}$$

Equation stabilizing constraint

$$\begin{aligned} \ddot{\mathbf{R}} + 2\alpha\dot{\mathbf{R}} + \alpha^2\mathbf{R} &= \mathbf{0} \\ \Downarrow \\ \begin{bmatrix} \boldsymbol{\omega}_{1,2}^\top Q_{1,2;x} \boldsymbol{\omega}_{1,2} \\ \boldsymbol{\omega}_{1,2}^\top Q_{1,2;y} \boldsymbol{\omega}_{1,2} \end{bmatrix} + J_{1,2}\dot{\boldsymbol{\omega}}_{1,2} - \begin{bmatrix} \boldsymbol{\omega}_{3,4}^\top Q_{3,4;x} \boldsymbol{\omega}_{3,4} \\ \boldsymbol{\omega}_{3,4}^\top Q_{3,4;y} \boldsymbol{\omega}_{3,4} \end{bmatrix} - J_{3,4}\dot{\boldsymbol{\omega}}_{3,4} \\ + 2\alpha(J_{1,2}\boldsymbol{\omega}_{1,2} - J_{3,4}\boldsymbol{\omega}_{3,4}) + \alpha^2\mathbf{R} &= \mathbf{0} \\ \Downarrow \end{aligned}$$

PD control

$$\begin{aligned} \tau_1 &= -K_{P1}(\theta_1 - \theta_1^d) - K_{D1}\dot{\theta}_1 \\ \tau_3 &= -K_{P3}(\theta_3 - \theta_3^d) - K_{D3}\dot{\theta}_3 \end{aligned}$$

Sample Programs

- class **Closed_Mechanism_Two_DOF**
- [closed_mechanism_2DOF_PD.m](#)
PD control of 2DOF closed mechanism
- [closed_mechanism_2DOF_PD_params.m](#) equation of motion

Equation stabilizing constraint

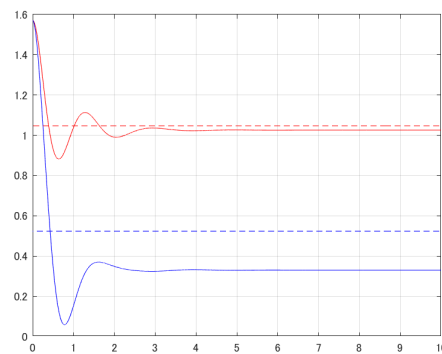
$$\begin{bmatrix} -J_{1,2} & J_{3,4} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \end{bmatrix} = \mathbf{C}$$

where

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} \boldsymbol{\omega}_{1,2}^\top Q_{1,2;x} \boldsymbol{\omega}_{1,2} \\ \boldsymbol{\omega}_{1,2}^\top Q_{1,2;y} \boldsymbol{\omega}_{1,2} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\omega}_{3,4}^\top Q_{3,4;x} \boldsymbol{\omega}_{3,4} \\ \boldsymbol{\omega}_{3,4}^\top Q_{3,4;y} \boldsymbol{\omega}_{3,4} \end{bmatrix} \\ &\quad + 2\alpha(J_{1,2}\boldsymbol{\omega}_{1,2} - J_{3,4}\boldsymbol{\omega}_{3,4}) + \alpha^2\mathbf{R} \end{aligned}$$

PD control

Result



Dynamic equations for closed link mechanism

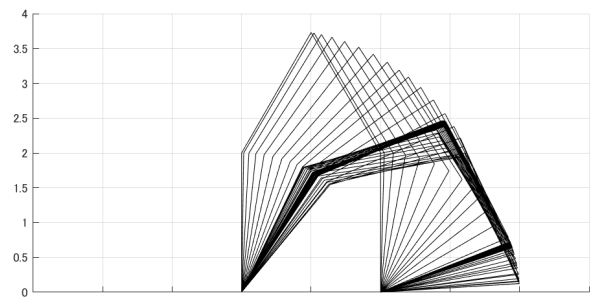
Combining Lagrange equation of motion and equation stabilizing constraint yields

$$\begin{bmatrix} H_{1,2} & O_{2 \times 2} & -J_{1,2}^\top \\ O_{2 \times 2} & H_{3,4} & J_{3,4}^\top \\ -J_{1,2} & J_{3,4} & O_{2 \times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\omega}}_{1,2} \\ \dot{\boldsymbol{\omega}}_{3,4} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{1,2} + \boldsymbol{\tau}_{left} \\ \boldsymbol{\tau}_{3,4} + \boldsymbol{\tau}_{right} \\ \mathbf{C} \end{bmatrix}$$

coefficient matrix is regular $\rightarrow$ we can compute $\dot{\boldsymbol{\omega}}_1$ through $\dot{\boldsymbol{\omega}}_4$

PD control

Result



Physical Interpretation

$J_{1,2}$ and $J_{3,4}$: Jacobian matrices of the left and right arms
 $\boldsymbol{\lambda} = [\lambda_x, \lambda_y]^\top$: constraint force
 equivalent torques around rotational joints 1 and 2:

$$J_{1,2}^\top \boldsymbol{\lambda} = \begin{bmatrix} \lambda_x(-l_1 S_1 - l_2 S_{1+2}) + \lambda_y(l_1 C_1 + l_2 C_{1+2}) \\ \lambda_x(-l_2 S_{1+2}) + \lambda_y l_2 C_{1+2} \end{bmatrix}$$

reaction force $-\boldsymbol{\lambda}$

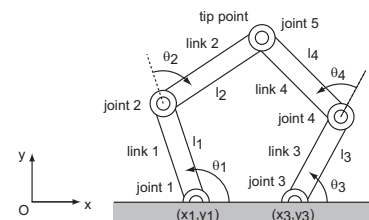
equivalent torques around rotational joint 3 and 4:

$$J_{3,4}^\top (-\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_x(l_3 S_3 + l_4 S_{3+4}) + \lambda_y(-l_3 C_3 - l_4 C_{3+4}) \\ \lambda_x l_4 S_{3+4} + \lambda_y(-l_4 C_{3+4}) \end{bmatrix}$$

Report

Report #4 due date : Nov. 25 (Mon) 1:00 AM

Simulate the motion of a 2DOF closed link mechanism under PID control. PID control is applied to active joints 1 and 3. Use appropriate values of geometrical and physical parameters of the manipulator.



Summary

Open link mechanism

- inertia matrix depends on joint angles
- Lagrange equations of motion of open link mechanism

Closed link mechanism

- two open link mechanisms with geometric constraints
- synthesized from Lagrange equations of open link mechanisms