Analytical Mechanics: Rigid Body Rotation

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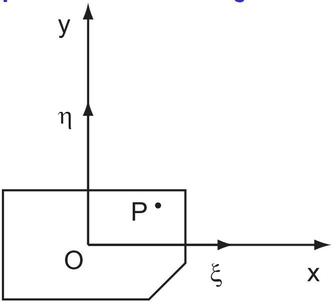
Agenda

Spatial and object coordinate systems



$O - xy$	spatial coordinate system
$O - \xi\eta$	object coordinate system
$\boldsymbol{a}$	unit vector along ξ-axis
$\boldsymbol{b}$	unit vector along η-axis

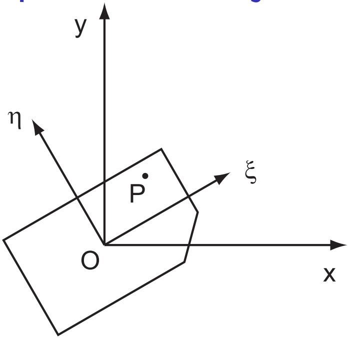
$$\| \boldsymbol{a} \| = 1 \quad \rightarrow \quad \boldsymbol{a}^\top \boldsymbol{a} = 1$$
$$\| \boldsymbol{b} \| = 1 \quad \rightarrow \quad \boldsymbol{b}^\top \boldsymbol{b} = 1$$
$$\boldsymbol{a} \perp \boldsymbol{b} \quad \rightarrow \quad \boldsymbol{a}^\top \boldsymbol{b} = \boldsymbol{b}^\top \boldsymbol{a} = 0$$

ξ, η: object coordinates of point P spatial coordinates of point P

$$\boldsymbol{x} = \xi \boldsymbol{a} + \eta \boldsymbol{b} = \underbrace{\left[\begin{array}{c|c} \boldsymbol{a} & \boldsymbol{b} \end{array} \right]}_{R} \underbrace{\left[\begin{array}{c} \xi \\ \eta \end{array} \right]}_{\boldsymbol{\xi}}$$

Note: $\boldsymbol{a}$ and $\boldsymbol{b}$ depend on time. ξ and η are independent of time.

Spatial and object coordinate systems



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Note: $\boldsymbol{a}$ and $\boldsymbol{b}$ depend on time. ξ and η are independent of time.

Angular velocity in planar rotation

differentiating relationships between $\boldsymbol{a}$ and $\boldsymbol{b}$ w.r.t time:

$$\boldsymbol{a}^\top \boldsymbol{a} = 1 \rightarrow \boldsymbol{a}^\top \dot{\boldsymbol{a}} = 0$$

$$\boldsymbol{b}^\top \boldsymbol{b} = 1 \rightarrow \boldsymbol{b}^\top \dot{\boldsymbol{b}} = 0$$

$$\boldsymbol{a}^\top \boldsymbol{b} = 0 \rightarrow \underbrace{\boldsymbol{a}^\top \dot{\boldsymbol{b}}}_{-\omega} + \underbrace{\boldsymbol{b}^\top \dot{\boldsymbol{a}}}_{\omega} = 0$$

describing $\dot{\boldsymbol{a}}$ and $\dot{\boldsymbol{b}}$ in object coordinate system:

$$\dot{\boldsymbol{a}} = (\boldsymbol{a}^\top \dot{\boldsymbol{a}})\boldsymbol{a} + (\boldsymbol{b}^\top \dot{\boldsymbol{a}})\boldsymbol{b} \; = \quad \omega \boldsymbol{b}$$

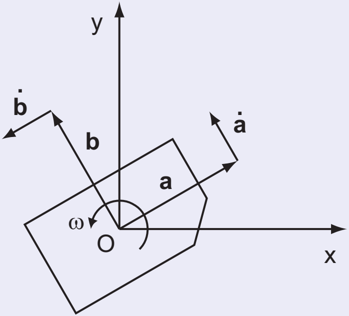
$$\dot{\boldsymbol{b}} = (\boldsymbol{a}^\top \dot{\boldsymbol{b}})\boldsymbol{a} + (\boldsymbol{b}^\top \dot{\boldsymbol{b}})\boldsymbol{b} \; = \; -\omega \boldsymbol{a}$$

velocity of point $\mathrm{P}(\xi, \eta)$

$$\dot{\boldsymbol{x}} = \xi \dot{\boldsymbol{a}} + \eta \dot{\boldsymbol{b}} = \omega(\xi \boldsymbol{b} - \eta \boldsymbol{a})$$

Angular velocity in planar rotation

interpretation



$\omega = \boldsymbol{b}^{\top} \dot{\boldsymbol{a}}$ angular velocity

$\| \boldsymbol{a} \| = 1 \quad \rightarrow \quad \dot{\boldsymbol{a}} \perp \boldsymbol{a} \rightarrow \quad \dot{\boldsymbol{a}} \parallel \boldsymbol{b}$

$\| \boldsymbol{b} \| = 1 \quad \rightarrow \quad \dot{\boldsymbol{b}} \perp \boldsymbol{b} \rightarrow \quad \dot{\boldsymbol{b}} \parallel \boldsymbol{a}$

$\| \boldsymbol{a} \| = 1 \quad \rightarrow \quad \| \dot{\boldsymbol{a}} \| = \omega$

$\| \boldsymbol{a} \| = 1 \quad \rightarrow \quad \| \dot{\boldsymbol{b}} \| = \omega$

Kinetic energy of rigid body

Divide a rigid body into a finite number of masses.

m_i	the i-th mass
(ξ_i, η_i)	object coordinates of the i-th mass

position of mass m_i $\quad \boldsymbol{x}_i = \xi_i \boldsymbol{a} + \eta_i \boldsymbol{b}$

velocity of mass m_i $\quad \dot{\boldsymbol{x}}_i = \xi_i \dot{\boldsymbol{a}} + \eta_i \dot{\boldsymbol{b}} = \omega(\xi_i \boldsymbol{b} - \eta_i \boldsymbol{a})$

kinetic energy of mass m_i

$$\begin{aligned} \frac{1}{2} m_i \dot{\boldsymbol{x}}_i^\top \dot{\boldsymbol{x}}_i &= \frac{1}{2} m_i \, \omega^2 (\xi_i \boldsymbol{b} - \eta_i \boldsymbol{a})^\top (\xi_i \boldsymbol{b} - \eta_i \boldsymbol{a}) \\ &= \frac{1}{2} m_i \, \omega^2 (\xi_i^2 + \eta_i^2) \end{aligned}$$

Kinetic energy of rigid body

kinetic energy of rigid body rotating on plane

$$\sum_i \frac{1}{2} m_i \dot{\boldsymbol{x}}_i^\top \dot{\boldsymbol{x}}_i = \sum_i \frac{1}{2} m_i \, \omega^2 (\xi_i^2 + \eta_i^2) = \frac{1}{2} J \omega^2$$

where

$$J = \sum_i m_i \, (\xi_i^2 + \eta_i^2)$$

inertia of moment

Note: J is constant (independent of time)

Computing Lagrange equation of planar rotation

description that satisfies relationships between $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{a} = \begin{bmatrix} C_\theta \\ S_\theta \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} -S_\theta \\ C_\theta \end{bmatrix}$$

angular velocity

$$\omega = \boldsymbol{b}^\top \dot{\boldsymbol{a}} = \begin{bmatrix} -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} -S_\theta \\ C_\theta \end{bmatrix} \dot{\theta} = \dot{\theta}$$

kinetic energy

$$T = \frac{1}{2} J \dot{\theta}^2$$

work done by external torque τ around point O

$$W = \tau\theta$$

Computing Lagrange equation of planar rotation

Lagrangian

$$L = \frac{1}{2} J \dot{\theta}^2 + \tau \theta$$

Lagrange equation of motion

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\tau - J \ddot{\theta} = 0$$

$$J \ddot{\theta} = \tau$$

equation of planar rotation

Inertia of moment in rigid continuum

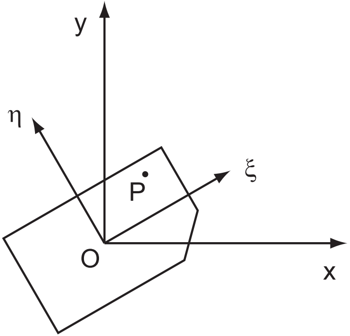
Let ρ be planar density of a rigid body.

$$m_i \to \rho \, \mathrm{d}\xi \mathrm{d}\eta$$
$$\sum_i \to \int_S$$

inertia of moment:

$$J = \sum_i m_i \left(\xi_i^2 + \eta_i^2\right)$$
$$\downarrow$$
$$J = \int_S \rho \left(\xi^2 + \eta^2\right) \mathrm{d}\xi \mathrm{d}\eta$$

Introducing rotation matrix



spatial coordinates of point $\mathrm{P}(\xi, \eta)$

$$\begin{aligned} \boldsymbol{x} &= \xi \boldsymbol{a} + \eta \boldsymbol{b} \\ &= \underbrace{\left[\begin{array}{c|c} \boldsymbol{a} & \boldsymbol{b} \end{array} \right]}_{R} \underbrace{\left[\begin{array}{c} \xi \\ \eta \end{array} \right]}_{\boldsymbol{\xi}} \\ &= R\boldsymbol{\xi} \end{aligned}$$

R rotation matrix

$$R^\top R = \left[\begin{array}{c} \boldsymbol{a}^\top \\ \boldsymbol{b}^\top \end{array} \right] \left[\begin{array}{c|c} \boldsymbol{a} & \boldsymbol{b} \end{array} \right] = \left[\begin{array}{cc} \boldsymbol{a}^\top \boldsymbol{a} & \boldsymbol{a}^\top \boldsymbol{b} \\ \boldsymbol{b}^\top \boldsymbol{a} & \boldsymbol{b}^\top \boldsymbol{b} \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

R is an orthogonal matrix

$$R^\top R = RR^\top = I_{2\times 2} \quad \text{(unit matrix)}$$

Computing kinetic energy using rotation matrix

differentiating $R^\top R = I_{2\times 2}$ w.r.t time:

$$\dot{R}^\top R + R^\top \dot{R} = O_{2\times 2} \quad \text{(zero matrix)}$$

$$(R^\top \dot{R})^\top + (R^\top \dot{R}) = O_{2\times 2}$$

$R^\top \dot{R}$ is a skew-symmetric matrix:

$$R^\top \dot{R} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

$$\begin{aligned}(R^\top \dot{R})^\top (R^\top \dot{R}) &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \\ &= \omega^2 \, I_{2\times 2}\end{aligned}$$

Computing kinetic energy using rotation matrix

differentiating $\boldsymbol{x} = R\boldsymbol{\xi}$ with respect to time:

$$\dot{\boldsymbol{x}} = \dot{R}\boldsymbol{\xi}$$

quadratic form:

$$\begin{aligned}\dot{\boldsymbol{x}}^\top \dot{\boldsymbol{x}} &= \boldsymbol{\xi}^\top \dot{R}^\top \dot{R} \boldsymbol{\xi} \;=\; \boldsymbol{\xi}^\top \dot{R}^\top R R^\top \dot{R} \boldsymbol{\xi} \\ &= \boldsymbol{\xi}^\top (R^\top \dot{R})^\top (R^\top \dot{R})\, \boldsymbol{\xi} \;=\; \boldsymbol{\xi}^\top \omega^2 I_{2\times 2}\, \boldsymbol{\xi} \\ &= \omega^2\, \boldsymbol{\xi}^\top \boldsymbol{\xi}\end{aligned}$$

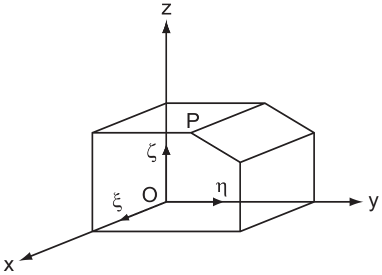
kinetic energy of a rigid body:

$$T = \sum_i \frac{1}{2} m_i\, \dot{\boldsymbol{x}}_i^\top \dot{\boldsymbol{x}}_i \;=\; \sum_i \frac{1}{2} m_i\, \omega^2\, \boldsymbol{\xi}_i^\top \boldsymbol{\xi}_i \;=\; \frac{1}{2} J \omega^2$$

where

$$J = \sum_i m_i\, \boldsymbol{\xi}_i^\top \boldsymbol{\xi}_i = \sum_i m_i\, (\xi_i^2 + \eta_i^2)$$

Spatial and object coordinate systems



$O - xyz$ spatial coordinate system
$O - \xi\eta\zeta$ object coordinate system

$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ unit vectors along ξ-, η-, and ζ axes

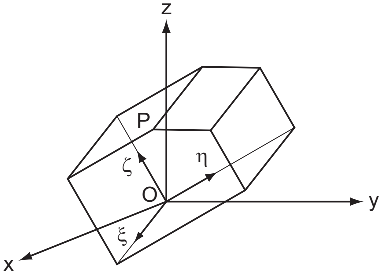
$$\boldsymbol{a}^\top\boldsymbol{a} = \boldsymbol{b}^\top\boldsymbol{b} = \boldsymbol{c}^\top\boldsymbol{c} = 1$$
$$\boldsymbol{a}^\top\boldsymbol{b} = \boldsymbol{b}^\top\boldsymbol{c} = \boldsymbol{c}^\top\boldsymbol{a} = 0$$

ξ, η, ζ: object coordinates of point P spatial coordinates of point P

$$\boldsymbol{x} = \xi\boldsymbol{a} + \eta\boldsymbol{b} + \zeta\boldsymbol{c} = \underbrace{\left[\begin{array}{c|c|c} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{array}\right]}_{R} \underbrace{\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}}_{\boldsymbol{\xi}} = R\boldsymbol{\xi}$$

Note: $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ depend on time. ξ, η, ζ are independent of time.

Spatial and object coordinate systems



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ξ, η, ζ: object coordinates of point P spatial coordinates of point P

$$\boldsymbol{x} = \xi \boldsymbol{a} + \eta \boldsymbol{b} + \zeta \boldsymbol{c} = \underbrace{\left[\begin{array}{c|c|c} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{array} \right]}_{R} \underbrace{\left[\begin{array}{c} \xi \\ \eta \\ \zeta \end{array} \right]}_{\boldsymbol{\xi}} = R\boldsymbol{\xi}$$

Note: $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ depend on time. ξ, η, ζ are independent of time.

Angular velocity vector in spatial rotation

rotation matrix R

$$R^\top R = \begin{bmatrix} \boldsymbol{a}^\top \\ \boldsymbol{b}^\top \\ \boldsymbol{c}^\top \end{bmatrix} \left[\begin{array}{c|c|c} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{array} \right] = I_{3\times 3} \quad \text{(unit matrix)}$$

R is an orthogonal matrix

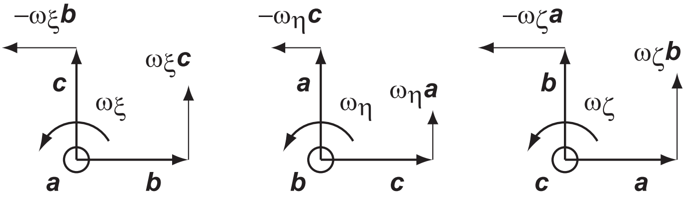
differentiating $R^\top R = I_{3\times 3}$ w.r.t time:

$$\dot{R}^\top R + R^\top \dot{R} = (R^\top \dot{R})^\top + (R^\top \dot{R}) = O_{3\times 3} \quad \text{(zero matrix)}$$

$R^\top \dot{R}$ is a skew-symmetric matrix:

$$R^\top \dot{R} = \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix}$$

Angular velocity vector in spatial rotation



$\omega_\xi = \boldsymbol{c}^\top \dot{\boldsymbol{b}}$ $\omega_\eta = \boldsymbol{a}^\top \dot{\boldsymbol{c}}$ $\omega_\zeta = \boldsymbol{b}^\top \dot{\boldsymbol{a}}$

$$R^\top \dot{R} = \begin{bmatrix} \boldsymbol{a}^\top \\ \boldsymbol{b}^\top \\ \boldsymbol{c}^\top \end{bmatrix} \left[\begin{array}{c|c|c} \dot{\boldsymbol{a}} & \dot{\boldsymbol{b}} & \dot{\boldsymbol{c}} \end{array} \right] = \begin{bmatrix} \boldsymbol{a}^\top \dot{\boldsymbol{a}} & \boldsymbol{a}^\top \dot{\boldsymbol{b}} & \boldsymbol{a}^\top \dot{\boldsymbol{c}} \\ \boldsymbol{b}^\top \dot{\boldsymbol{a}} & \boldsymbol{b}^\top \dot{\boldsymbol{b}} & \boldsymbol{b}^\top \dot{\boldsymbol{c}} \\ \boldsymbol{c}^\top \dot{\boldsymbol{a}} & \boldsymbol{c}^\top \dot{\boldsymbol{b}} & \boldsymbol{c}^\top \dot{\boldsymbol{c}} \end{bmatrix}$$

$$\dot{\boldsymbol{a}} = (\boldsymbol{a}^\top \dot{\boldsymbol{a}})\boldsymbol{a} + (\boldsymbol{b}^\top \dot{\boldsymbol{a}})\boldsymbol{b} + (\boldsymbol{c}^\top \dot{\boldsymbol{a}})\boldsymbol{c} = \omega_\zeta \boldsymbol{b} - \omega_\eta \boldsymbol{c}$$

$$\dot{\boldsymbol{b}} = (\boldsymbol{a}^\top \dot{\boldsymbol{b}})\boldsymbol{a} + (\boldsymbol{b}^\top \dot{\boldsymbol{b}})\boldsymbol{b} + (\boldsymbol{c}^\top \dot{\boldsymbol{b}})\boldsymbol{c} = \omega_\xi \boldsymbol{c} - \omega_\zeta \boldsymbol{a}$$

$$\dot{\boldsymbol{c}} = (\boldsymbol{a}^\top \dot{\boldsymbol{c}})\boldsymbol{a} + (\boldsymbol{b}^\top \dot{\boldsymbol{c}})\boldsymbol{b} + (\boldsymbol{c}^\top \dot{\boldsymbol{c}})\boldsymbol{c} = \omega_\eta \boldsymbol{a} - \omega_\xi \boldsymbol{b}$$

Angular velocity vector in spatial rotation

angular velocity vector:

$$\boldsymbol{\omega} \triangleq \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}$$

Note that $\boldsymbol{\omega}$ is defined on object coordinate system.

$$\begin{aligned}(R^\top \dot{R})\,\boldsymbol{\xi} &= \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \omega_\eta \zeta - \omega_\zeta \eta \\ \omega_\zeta \xi - \omega_\xi \zeta \\ \omega_\xi \eta - \omega_\eta \xi \end{bmatrix} \\ &= \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix} \times \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \boldsymbol{\omega} \times \boldsymbol{\xi}\end{aligned}$$

Computing kinetic energy in spacial rotation

differentiating $\boldsymbol{x} = R\boldsymbol{\xi}$ with respect to time:

$$\dot{\boldsymbol{x}} = \dot{R}\boldsymbol{\xi}$$

quadratic form:

$$\begin{aligned}
\dot{\boldsymbol{x}}^\top \dot{\boldsymbol{x}} &= \boldsymbol{\xi}^\top \dot{R}^\top \dot{R}\boldsymbol{\xi} \;=\; \boldsymbol{\xi}^\top \dot{R}^\top R R^\top \dot{R}\boldsymbol{\xi} \\
&= (\boldsymbol{\omega} \times \boldsymbol{\xi})^\top (\boldsymbol{\omega} \times \boldsymbol{\xi}) \;=\; (-\boldsymbol{\xi} \times \boldsymbol{\omega})^\top (-\boldsymbol{\xi} \times \boldsymbol{\omega}) \\
&= (\boldsymbol{\xi} \times \boldsymbol{\omega})^\top (\boldsymbol{\xi} \times \boldsymbol{\omega}) \;=\; ([\,\boldsymbol{\xi}\times\,]\boldsymbol{\omega})^\top ([\,\boldsymbol{\xi}\times\,]\boldsymbol{\omega}) \\
&= \boldsymbol{\omega}^\top \,[\,\boldsymbol{\xi}\times\,]^\top [\,\boldsymbol{\xi}\times\,]\,\boldsymbol{\omega}
\end{aligned}$$

where

$$[\,\boldsymbol{\xi}\times\,] \triangleq \begin{bmatrix} 0 & -\zeta & \eta \\ \zeta & 0 & -\xi \\ -\eta & \xi & 0 \end{bmatrix}$$

Computing kinetic energy in spacial rotation

$$
[\ \boldsymbol{\xi}\times\ ]^\top [\ \boldsymbol{\xi}\times\ ] = \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \begin{bmatrix} 0 & -\zeta & \eta \\ \zeta & 0 & -\xi \\ -\eta & \xi & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} \eta^2 + \zeta^2 & -\xi\eta & -\xi\zeta \\ -\eta\xi & \zeta^2 + \xi^2 & -\eta\zeta \\ -\zeta\xi & -\zeta\eta & \xi^2 + \eta^2 \end{bmatrix}
$$

kinetic energy of a rigid body:

$$
T = \sum_i \frac{1}{2} m_i \, \dot{\boldsymbol{x}}_i^\top \dot{\boldsymbol{x}}_i \ = \ \sum_i \frac{1}{2} m_i \, \boldsymbol{\omega}^\top \ [\ \boldsymbol{\xi}_i\times\ ]^\top [\ \boldsymbol{\xi}_i\times\ ] \ \boldsymbol{\omega}
$$

$$
= \frac{1}{2} \boldsymbol{\omega}^\top J \, \boldsymbol{\omega}
$$

Computing kinetic energy in spacial rotation

inertia matrix

$$J \triangleq \sum_i m_i \left[\boldsymbol{\xi}_i \times \right]^\top \left[\boldsymbol{\xi}_i \times \right] = \begin{bmatrix} J_\xi & J_{\xi\eta} & J_{\zeta\xi} \\ J_{\xi\eta} & J_\eta & J_{\eta\zeta} \\ J_{\zeta\xi} & J_{\eta\zeta} & J_\zeta \end{bmatrix}$$

where

$$J_\xi = \sum_i m_i \, (\eta_i^2 + \zeta_i^2), \quad J_\eta = \sum_i m_i \, (\zeta_i^2 + \xi_i^2),$$

$$J_\zeta = \sum_i m_i \, (\xi_i^2 + \eta_i^2),$$

$$J_{\xi\eta} = -\sum_i m_i \, \xi_i \eta_i, \quad J_{\eta\zeta} = -\sum_i m_i \, \eta_i \zeta_i, \quad J_{\zeta\xi} = -\sum_i m_i \, \zeta_i \xi_i$$

Note: inertia matrix is constant (independent of time)

Lagrange equation of spatial rotation

generalized coordinates describing spatial rotation:

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

under geometric constraints:

$$R_1 = \boldsymbol{a}^\top \boldsymbol{a} - 1 = 0, \quad R_2 = \boldsymbol{b}^\top \boldsymbol{b} - 1 = 0, \quad R_3 = \boldsymbol{c}^\top \boldsymbol{c} - 1 = 0,$$
$$Q_1 = \boldsymbol{b}^\top \boldsymbol{c} = 0, \quad Q_2 = \boldsymbol{c}^\top \boldsymbol{a} = 0, \quad Q_3 = \boldsymbol{a}^\top \boldsymbol{b} = 0$$

kinetic energy

$$T = \frac{1}{2} \boldsymbol{\omega}^\top J \, \boldsymbol{\omega}$$

where

$$\omega_\xi = \boldsymbol{c}^\top \dot{\boldsymbol{b}}, \quad \omega_\eta = \boldsymbol{a}^\top \dot{\boldsymbol{c}}, \quad \omega_\zeta = \boldsymbol{b}^\top \dot{\boldsymbol{a}}$$

Lagrange equation of spatial rotation

Lagrangian

$$L = T + \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 + \mu_1 Q_1 + \mu_2 Q_2 + \mu_3 Q_3$$

(λ_1, λ_2, λ_3, μ_1, μ_2, μ_3: Lagrange multipliers)

Lagrange equations of spacial rotation

$$\frac{\partial L}{\partial \boldsymbol{a}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{a}}} = \mathbf{0},$$
$$\frac{\partial L}{\partial \boldsymbol{b}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{b}}} = \mathbf{0},$$
$$\frac{\partial L}{\partial \boldsymbol{c}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{c}}} = \mathbf{0}$$

Computing Lagrange equation of spatial rotation

kinetic energy

$$T = \frac{1}{2}\,\boldsymbol{\omega}^\top J\,\boldsymbol{\omega}$$

derivative

$$\frac{\mathrm{d}\,T}{\mathrm{d}\boldsymbol{\omega}} = J\boldsymbol{\omega}$$

$$\frac{\mathrm{d}\,T}{\mathrm{d}\boldsymbol{\omega}} = \begin{bmatrix} \dfrac{\partial T}{\partial \omega_\xi} \\ \dfrac{\partial T}{\partial \omega_\eta} \\ \dfrac{\partial T}{\partial \omega_\zeta} \end{bmatrix}$$

Computing Lagrange equation of spatial rotation

angular velocities

$$\omega_\xi = \boldsymbol{c}^\top \dot{\boldsymbol{b}}, \quad \omega_\eta = \boldsymbol{a}^\top \dot{\boldsymbol{c}}, \quad \omega_\zeta = \boldsymbol{b}^\top \dot{\boldsymbol{a}}$$

partial derivatives

$$\frac{\partial \omega_\xi}{\partial \boldsymbol{a}} = \boldsymbol{0}, \quad \frac{\partial \omega_\eta}{\partial \boldsymbol{a}} = \dot{\boldsymbol{c}}, \quad \frac{\partial \omega_\zeta}{\partial \boldsymbol{a}} = \boldsymbol{0}$$
$$\frac{\partial \omega_\xi}{\partial \dot{\boldsymbol{a}}} = \boldsymbol{0}, \quad \frac{\partial \omega_\eta}{\partial \dot{\boldsymbol{a}}} = \boldsymbol{0}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\boldsymbol{a}}} = \boldsymbol{b}$$

Computing Lagrange equation of spatial rotation

angular velocities

$$\omega_\xi = \boldsymbol{c}^\top \dot{\boldsymbol{b}}, \quad \omega_\eta = \boldsymbol{a}^\top \dot{\boldsymbol{c}}, \quad \omega_\zeta = \boldsymbol{b}^\top \dot{\boldsymbol{a}}$$

partial derivatives

$$\frac{\partial \omega_\xi}{\partial \boldsymbol{a}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \boldsymbol{a}} = \dot{\boldsymbol{c}}, \quad \frac{\partial \omega_\zeta}{\partial \boldsymbol{a}} = \mathbf{0}, \quad \frac{\partial \omega_\xi}{\partial \dot{\boldsymbol{a}}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \dot{\boldsymbol{a}}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\boldsymbol{a}}} = \boldsymbol{b}$$

$$\frac{\partial \omega_\xi}{\partial \boldsymbol{b}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \boldsymbol{b}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \boldsymbol{b}} = \dot{\boldsymbol{a}}, \quad \frac{\partial \omega_\xi}{\partial \dot{\boldsymbol{b}}} = \boldsymbol{c}, \quad \frac{\partial \omega_\eta}{\partial \dot{\boldsymbol{b}}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\boldsymbol{b}}} = \mathbf{0}$$

$$\frac{\partial \omega_\xi}{\partial \boldsymbol{c}} = \dot{\boldsymbol{b}}, \quad \frac{\partial \omega_\eta}{\partial \boldsymbol{c}} = \mathbf{0}, \quad \frac{\partial \omega_\zeta}{\partial \boldsymbol{c}} = \mathbf{0}, \quad \frac{\partial \omega_\xi}{\partial \dot{\boldsymbol{c}}} = \mathbf{0}, \quad \frac{\partial \omega_\eta}{\partial \dot{\boldsymbol{c}}} = \boldsymbol{a}, \quad \frac{\partial \omega_\zeta}{\partial \dot{\boldsymbol{c}}} = \mathbf{0}$$

Computing Lagrange equation of spatial rotation

dependency

$$T \leftarrow \omega_\xi,\, \omega_\eta,\, \omega_\zeta \leftarrow \boldsymbol{a}$$

$$\Downarrow$$

$$\frac{\partial T}{\partial \boldsymbol{a}} = \frac{\partial T}{\partial \omega_\xi}\frac{\partial \omega_\xi}{\partial \boldsymbol{a}} + \frac{\partial T}{\partial \omega_\eta}\frac{\partial \omega_\eta}{\partial \boldsymbol{a}} + \frac{\partial T}{\partial \omega_\zeta}\frac{\partial \omega_\zeta}{\partial \boldsymbol{a}}$$

$$= \left[\begin{array}{c|c|c} \dfrac{\partial \omega_\xi}{\partial \boldsymbol{a}} & \dfrac{\partial \omega_\eta}{\partial \boldsymbol{a}} & \dfrac{\partial \omega_\zeta}{\partial \boldsymbol{a}} \end{array} \right] \begin{bmatrix} \dfrac{\partial T}{\partial \omega_\xi} \\ \dfrac{\partial T}{\partial \omega_\eta} \\ \dfrac{\partial T}{\partial \omega_\zeta} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \dot{\boldsymbol{c}} & \mathbf{0} \end{bmatrix} J \boldsymbol{\omega}$$

Computing Lagrange equation of spatial rotation

dependency

$$T \leftarrow \omega_\xi,\, \omega_\eta,\, \omega_\zeta \leftarrow \dot{\boldsymbol{a}}$$

$$\Downarrow$$

$$\begin{aligned}
\frac{\partial T}{\partial \dot{\boldsymbol{a}}} &= \frac{\partial T}{\partial \omega_\xi}\frac{\partial \omega_\xi}{\partial \dot{\boldsymbol{a}}} + \frac{\partial T}{\partial \omega_\eta}\frac{\partial \omega_\eta}{\partial \dot{\boldsymbol{a}}} + \frac{\partial T}{\partial \omega_\zeta}\frac{\partial \omega_\zeta}{\partial \dot{\boldsymbol{a}}} \\
&= \left[\begin{array}{c|c|c} \dfrac{\partial \omega_\xi}{\partial \dot{\boldsymbol{a}}} & \dfrac{\partial \omega_\eta}{\partial \dot{\boldsymbol{a}}} & \dfrac{\partial \omega_\zeta}{\partial \dot{\boldsymbol{a}}} \end{array} \right] \begin{bmatrix} \dfrac{\partial T}{\partial \omega_\xi} \\ \dfrac{\partial T}{\partial \omega_\eta} \\ \dfrac{\partial T}{\partial \omega_\zeta} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{bmatrix} J \boldsymbol{\omega}
\end{aligned}$$

Computing Lagrange equation of spatial rotation

partial derivative

$$\frac{\partial T}{\partial \dot{\boldsymbol{a}}} = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{array} \right] J \boldsymbol{\omega}$$

time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{\boldsymbol{a}}} = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{array} \right] J \dot{\boldsymbol{\omega}} + \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \dot{\boldsymbol{b}} \end{array} \right] J \boldsymbol{\omega}$$

Computing Lagrange equation of spatial rotation

geometric constraints

$$R_1 = \boldsymbol{a}^\top \boldsymbol{a} - 1 = 0, \quad R_2 = \boldsymbol{b}^\top \boldsymbol{b} - 1 = 0, \quad R_3 = \boldsymbol{c}^\top \boldsymbol{c} - 1 = 0,$$
$$Q_1 = \boldsymbol{b}^\top \boldsymbol{c} = 0, \quad Q_2 = \boldsymbol{c}^\top \boldsymbol{a} = 0, \quad Q_3 = \boldsymbol{a}^\top \boldsymbol{b} = 0$$

partial derivatives

$$\frac{\partial R_1}{\partial \boldsymbol{a}} = 2\boldsymbol{a}, \quad \frac{\partial R_2}{\partial \boldsymbol{a}} = \boldsymbol{0}, \quad \frac{\partial R_3}{\partial \boldsymbol{a}} = \boldsymbol{0}$$
$$\frac{\partial Q_1}{\partial \boldsymbol{a}} = \boldsymbol{0}, \quad \frac{\partial Q_2}{\partial \boldsymbol{a}} = \boldsymbol{c}, \quad \frac{\partial Q_3}{\partial \boldsymbol{a}} = \boldsymbol{b}$$

Computing Lagrange equation of spatial rotation

contributions to Lagrange equation of motion w.r.t. $\boldsymbol{a}$:

$$\begin{aligned}
\frac{\partial L}{\partial \boldsymbol{a}} &= \frac{\partial T}{\partial \boldsymbol{a}} + \lambda_1 \frac{\partial R_1}{\partial \boldsymbol{a}} + \cdots + \mu_3 \frac{\partial Q_3}{\partial \boldsymbol{a}} \\
&= \left[\begin{array}{ccc} \mathbf{0} & \dot{\boldsymbol{c}} & \mathbf{0} \end{array} \right] J\boldsymbol{\omega} + 2\lambda_1 \boldsymbol{a} + \mu_2 \boldsymbol{c} + \mu_3 \boldsymbol{b} \\
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{a}}} &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{\boldsymbol{a}}} \\
&= \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{array} \right] J\dot{\boldsymbol{\omega}} + \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \dot{\boldsymbol{b}} \end{array} \right] J\boldsymbol{\omega}
\end{aligned}$$

Lagrange equation of motion w.r.t. $\boldsymbol{a}$:

$$\begin{aligned}
\left[\begin{array}{ccc} \mathbf{0} & \dot{\boldsymbol{c}} & \mathbf{0} \end{array} \right] J\boldsymbol{\omega} - \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{array} \right] J\dot{\boldsymbol{\omega}} - \left[\begin{array}{ccc} \mathbf{0} & \mathbf{0} & \dot{\boldsymbol{b}} \end{array} \right] J\boldsymbol{\omega} \\
+ 2\lambda_1 \boldsymbol{a} + \mu_2 \boldsymbol{c} + \mu_3 \boldsymbol{b} = \mathbf{0}
\end{aligned}$$

Computing Lagrange equation of spatial rotation

Lagrange equation of motion w.r.t. $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \mathbf{0} & -\dot{\boldsymbol{c}} & \dot{\boldsymbol{b}} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_1 \boldsymbol{a} - \mu_2 \boldsymbol{c} - \mu_3 \boldsymbol{b} = \mathbf{0}$$

$$\begin{bmatrix} \boldsymbol{c} & \mathbf{0} & \mathbf{0} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \dot{\boldsymbol{c}} & \mathbf{0} & -\dot{\boldsymbol{a}} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_2 \boldsymbol{b} - \mu_3 \boldsymbol{a} - \mu_1 \boldsymbol{c} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{a} & \mathbf{0} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} -\dot{\boldsymbol{b}} & \dot{\boldsymbol{a}} & \mathbf{0} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_3 \boldsymbol{c} - \mu_1 \boldsymbol{b} - \mu_2 \boldsymbol{a} = \mathbf{0}$$

$\boldsymbol{c}^\top$(2nd eq.) (note: $\boldsymbol{c}^\top \boldsymbol{c} = 1$, $\boldsymbol{c}^\top \dot{\boldsymbol{c}} = 0$, and $-\boldsymbol{c}^\top \dot{\boldsymbol{a}} = \boldsymbol{a}^\top \dot{\boldsymbol{c}} = \omega_\eta$)

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & 0 & \omega_\eta \end{bmatrix} J\boldsymbol{\omega} - \mu_1 = 0$$

$\boldsymbol{b}^\top$(3rd eq.) (note: $\boldsymbol{b}^\top \boldsymbol{a} = 0$, $\boldsymbol{b}^\top \dot{\boldsymbol{b}} = 0$, and $\boldsymbol{b}^\top \dot{\boldsymbol{a}} = \omega_\zeta$)

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & \omega_\zeta & 0 \end{bmatrix} J\boldsymbol{\omega} - \mu_1 = 0$$

$\boldsymbol{c}^\top$(2nd eq.) $-$ $\boldsymbol{b}^\top$(3rd eq.)

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \end{bmatrix} J\boldsymbol{\omega} = 0$$

Computing Lagrange equation of spatial rotation

$\boldsymbol{c}^\top$(2nd eq.) $-$ $\boldsymbol{b}^\top$(3rd eq.)

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \end{bmatrix} J\boldsymbol{\omega} = 0$$

$\boldsymbol{a}^\top$(3rd eq.) $-$ $\boldsymbol{c}^\top$(1st eq.)

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \omega_\zeta & 0 & -\omega_\xi \end{bmatrix} J\boldsymbol{\omega} = 0$$

$\boldsymbol{b}^\top$(1st eq.) $-$ $\boldsymbol{a}^\top$(2nd eq.)

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} -\omega_\eta & \omega_\xi & 0 \end{bmatrix} J\boldsymbol{\omega} = 0$$

Euler's equation of rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} J\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J\dot{\boldsymbol{\omega}} + \left[\ \boldsymbol{\omega}\times\ \right] J\boldsymbol{\omega} = \mathbf{0}$$

Euler's equation of rotation

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J\boldsymbol{\omega} = \mathbf{0}$$

Dynamic equations describing spacial rotation

12 state variables

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

12 equations (6 differential eqs. and 6 algebraic eqs.)

$$\begin{aligned} & J\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times J\boldsymbol{\omega}, \\ & \boldsymbol{c}^\top \dot{\boldsymbol{b}} = \omega_\xi, \quad \boldsymbol{a}^\top \dot{\boldsymbol{c}} = \omega_\eta, \quad \boldsymbol{b}^\top \dot{\boldsymbol{a}} = \omega_\zeta, \\ & \boldsymbol{a}^\top \boldsymbol{a} = 1, \quad \boldsymbol{b}^\top \boldsymbol{b} = 1, \quad \boldsymbol{c}^\top \boldsymbol{c} = 1, \\ & \boldsymbol{a}^\top \boldsymbol{b} = 0, \quad \boldsymbol{b}^\top \boldsymbol{c} = 0, \quad \boldsymbol{c}^\top \boldsymbol{a} = 0 \end{aligned}$$

Lagrange equation of forced spatial rotation

external force $\boldsymbol{f}$ is applied to point $\mathrm{P}(\xi, \eta, \zeta)$:

$$W = \boldsymbol{f}^\top R\boldsymbol{\xi} \qquad (R\boldsymbol{\xi} \text{ denotes displacement vector of point } \mathrm{P})$$

$$\frac{\partial W}{\partial a_x} = \boldsymbol{f}^\top \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{f}^\top \begin{bmatrix} \xi \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi & 0 & 0 \end{bmatrix} \boldsymbol{f}$$

$$\frac{\partial W}{\partial a_y} = \boldsymbol{f}^\top \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{f}^\top \begin{bmatrix} 0 \\ \xi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \xi & 0 \end{bmatrix} \boldsymbol{f}$$

$$\frac{\partial W}{\partial a_z} = \boldsymbol{f}^\top \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{\xi} = \boldsymbol{f}^\top \begin{bmatrix} 0 \\ 0 \\ \xi \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi \end{bmatrix} \boldsymbol{f}$$

$$\Longrightarrow \qquad \frac{\partial W}{\partial \boldsymbol{a}} = \xi \boldsymbol{f}$$

Lagrange equation of forced spatial rotation

partial derivatives of W w.r.t. $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$:

$$\frac{\partial W}{\partial \boldsymbol{a}} = \xi \boldsymbol{f}, \quad \frac{\partial W}{\partial \boldsymbol{b}} = \eta \boldsymbol{f}, \quad \frac{\partial W}{\partial \boldsymbol{c}} = \zeta \boldsymbol{f}$$

Lagrangian

$$L = T + \lambda_1 R_1 + \lambda_2 R_2 + \lambda_3 R_3 + \mu_1 Q_1 + \mu_2 Q_2 + \mu_3 Q_3 + W$$

Lagrange equation of motion w.r.t. $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$:

$$\begin{aligned}
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{b} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \mathbf{0} & -\dot{\boldsymbol{c}} & \dot{\boldsymbol{b}} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_1 \boldsymbol{a} - \mu_2 \boldsymbol{c} - \mu_3 \boldsymbol{b} &= \xi \boldsymbol{f} \\
\begin{bmatrix} \boldsymbol{c} & \mathbf{0} & \mathbf{0} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} \dot{\boldsymbol{c}} & \mathbf{0} & -\dot{\boldsymbol{a}} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_2 \boldsymbol{b} - \mu_3 \boldsymbol{a} - \mu_1 \boldsymbol{c} &= \eta \boldsymbol{f} \\
\begin{bmatrix} \mathbf{0} & \boldsymbol{a} & \mathbf{0} \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} -\dot{\boldsymbol{b}} & \dot{\boldsymbol{a}} & \mathbf{0} \end{bmatrix} J\boldsymbol{\omega} - 2\lambda_3 \boldsymbol{c} - \mu_1 \boldsymbol{b} - \mu_2 \boldsymbol{a} &= \zeta \boldsymbol{f}
\end{aligned}$$

Lagrange equation of forced spatial rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} J\dot{\boldsymbol{\omega}} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} J\boldsymbol{\omega} = \begin{bmatrix} \eta \boldsymbol{c}^\top \boldsymbol{f} - \zeta \boldsymbol{b}^\top \boldsymbol{f} \\ \zeta \boldsymbol{a}^\top \boldsymbol{f} - \xi \boldsymbol{c}^\top \boldsymbol{f} \\ \xi \boldsymbol{b}^\top \boldsymbol{f} - \eta \boldsymbol{a}^\top \boldsymbol{f} \end{bmatrix}$$

external torque:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_\xi \\ \tau_\eta \\ \tau_\zeta \end{bmatrix} \triangleq \begin{bmatrix} \eta \boldsymbol{c}^\top \boldsymbol{f} - \zeta \boldsymbol{b}^\top \boldsymbol{f} \\ \zeta \boldsymbol{a}^\top \boldsymbol{f} - \xi \boldsymbol{c}^\top \boldsymbol{f} \\ \xi \boldsymbol{b}^\top \boldsymbol{f} - \eta \boldsymbol{a}^\top \boldsymbol{f} \end{bmatrix} = \begin{bmatrix} \eta f_\zeta - \zeta f_\eta \\ \zeta f_\xi - \xi f_\zeta \\ \xi f_\eta - \eta f_\xi \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \times \begin{bmatrix} f_\xi \\ f_\eta \\ f_\zeta \end{bmatrix}$$

Euler's equation of rotation with external torque

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times J\boldsymbol{\omega} = \boldsymbol{\tau}$$

Rotation matrix using quaternion

Definition of quaternion

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

where

$$\boldsymbol{q}^\top \boldsymbol{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Rotation matrix using quaternion

$$R(\boldsymbol{q}) = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

Describing column vectors of rotation matrix

column vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ of rotation matrix $R(\boldsymbol{q})$:

$$\boldsymbol{a} = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 \\ 2(q_1 q_2 + q_0 q_3) \\ 2(q_1 q_3 - q_0 q_2) \end{bmatrix} = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq A\boldsymbol{q}$$

$$\boldsymbol{b} = \begin{bmatrix} 2(q_1 q_2 - q_0 q_3) \\ 2(q_0^2 + q_2^2) - 1 \\ 2(q_2 q_3 + q_0 q_1) \end{bmatrix} = \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq B\boldsymbol{q}$$

$$\boldsymbol{c} = \begin{bmatrix} 2(q_1 q_3 + q_0 q_2) \\ 2(q_2 q_3 - q_0 q_1) \\ 2(q_0^2 + q_3^2) - 1 \end{bmatrix} = \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \triangleq C\boldsymbol{q}$$

Describing column vectors of rotation matrix

$$A^\top A = \begin{bmatrix} 1-q_1^2 & q_0 q_1 & q_1 q_3 & -q_1 q_2 \\ q_0 q_1 & 1-q_0^2 & -q_0 q_3 & q_0 q_2 \\ q_1 q_3 & -q_0 q_3 & 1-q_3^2 & q_2 q_3 \\ -q_1 q_2 & q_0 q_2 & q_2 q_3 & 1-q_2^2 \end{bmatrix}, \quad (A^\top A)\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$B^\top B = \begin{bmatrix} 1-q_2^2 & -q_2 q_3 & q_0 q_2 & q_1 q_2 \\ -q_2 q_3 & 1-q_3^2 & q_0 q_3 & q_1 q_3 \\ q_0 q_2 & q_0 q_3 & 1-q_0^2 & -q_0 q_1 \\ q_1 q_2 & q_1 q_3 & -q_0 q_1 & 1-q_1^2 \end{bmatrix}, \quad (B^\top B)\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$C^\top C = \begin{bmatrix} 1-q_3^2 & q_2 q_3 & -q_1 q_3 & q_0 q_3 \\ q_2 q_3 & 1-q_2^2 & q_1 q_2 & -q_0 q_2 \\ -q_1 q_3 & q_1 q_2 & 1-q_1^2 & q_0 q_1 \\ q_0 q_3 & -q_0 q_2 & q_0 q_1 & 1-q_0^2 \end{bmatrix}, \quad (C^\top C)\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Describing column vectors of rotation matrix

$$A^\top B = \begin{bmatrix} -q_1 q_2 & -q_1 q_3 & q_0 q_1 & q_1^2 - 1 \\ q_0 q_2 & q_0 q_3 & 1 - q_0^2 & -q_0 q_1 \\ q_2 q_3 & q_3^2 - 1 & -q_0 q_3 & -q_1 q_3 \\ 1 - q_2^2 & -q_2 q_3 & q_0 q_2 & q_1 q_2 \end{bmatrix}, \quad (A^\top B)\boldsymbol{q} = \begin{bmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{bmatrix}$$

$$B^\top C = \begin{bmatrix} -q_2 q_3 & q_2^2 - 1 & -q_1 q_2 & q_0 q_2 \\ 1 - q_3^2 & q_2 q_3 & -q_1 q_3 & q_0 q_3 \\ q_0 q_3 & -q_0 q_2 & q_0 q_1 & 1 - q_0^2 \\ q_1 q_3 & -q_1 q_2 & q_1^2 - 1 & -q_0 q_1 \end{bmatrix}, \quad (B^\top C)\boldsymbol{q} = \begin{bmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{bmatrix}$$

$$C^\top A = \begin{bmatrix} -q_1 q_3 & q_0 q_3 & q_3^2 - 1 & -q_2 q_3 \\ q_1 q_2 & -q_0 q_2 & -q_2 q_3 & q_2^2 - 1 \\ 1 - q_1^2 & q_0 q_1 & q_1 q_3 & -q_1 q_2 \\ q_0 q_1 & 1 - q_0^2 & -q_0 q_3 & q_0 q_2 \end{bmatrix}, \quad (C^\top A)\boldsymbol{q} = \begin{bmatrix} -q_2 \\ -q_3 \\ q_0 \\ q_1 \end{bmatrix}$$

Describing column vectors of rotation matrix

$$\boldsymbol{a}^\top \boldsymbol{a} = (A\boldsymbol{q})^\top (A\boldsymbol{q}) = \boldsymbol{q}^\top A^\top A \boldsymbol{q} = \boldsymbol{q}^\top \boldsymbol{q} = 1$$

$$\boldsymbol{b}^\top \boldsymbol{b} = (B\boldsymbol{q})^\top (B\boldsymbol{q}) = \boldsymbol{q}^\top B^\top B \boldsymbol{q} = \boldsymbol{q}^\top \boldsymbol{q} = 1$$

$$\cdots$$

$$\boldsymbol{a}^\top \boldsymbol{b} = (A\boldsymbol{q})^\top (B\boldsymbol{q}) = \boldsymbol{q}^\top A^\top B \boldsymbol{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} -q_3 \\ q_2 \\ -q_1 \\ q_0 \end{bmatrix} = 0$$

$$\boldsymbol{b}^\top \boldsymbol{c} = (B\boldsymbol{q})^\top (C\boldsymbol{q}) = \boldsymbol{q}^\top B^\top C \boldsymbol{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} -q_1 \\ q_0 \\ q_3 \\ -q_2 \end{bmatrix} = 0$$

$$\cdots$$

Describing column vectors of rotation matrix

Assume that $\boldsymbol{q}$ depends on time.
time derivative of quaternion:

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Note that matrices $R(\boldsymbol{q})$, A, B, and C depend on time.

Describing angular velocity vector using quaternion

time derivatives of column vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$:

$$\begin{aligned}
\dot{\boldsymbol{a}} &= \dot{A}\boldsymbol{q} + A\dot{\boldsymbol{q}} = A\dot{\boldsymbol{q}} + A\dot{\boldsymbol{q}} \;=\; 2A\dot{\boldsymbol{q}} \\
\dot{\boldsymbol{b}} &= \dot{B}\boldsymbol{q} + B\dot{\boldsymbol{q}} = B\dot{\boldsymbol{q}} + B\dot{\boldsymbol{q}} \;=\; 2B\dot{\boldsymbol{q}} \\
\dot{\boldsymbol{c}} &= \dot{C}\boldsymbol{q} + C\dot{\boldsymbol{q}} = C\dot{\boldsymbol{q}} + C\dot{\boldsymbol{q}} \;=\; 2C\dot{\boldsymbol{q}}
\end{aligned}$$

angular velocities:

$$\begin{aligned}
\omega_\xi &= \boldsymbol{c}^\top \dot{\boldsymbol{b}} \;=\; \boldsymbol{q}^\top C^\top 2B\dot{\boldsymbol{q}} \;=\; 2(B^\top C\boldsymbol{q})^\top \dot{\boldsymbol{q}} \\
\omega_\eta &= \boldsymbol{a}^\top \dot{\boldsymbol{c}} \;=\; \boldsymbol{q}^\top A^\top 2C\dot{\boldsymbol{q}} \;=\; 2(C^\top A\boldsymbol{q})^\top \dot{\boldsymbol{q}} \\
\omega_\zeta &= \boldsymbol{b}^\top \dot{\boldsymbol{a}} \;=\; \boldsymbol{q}^\top B^\top 2A\dot{\boldsymbol{q}} \;=\; 2(A^\top B\boldsymbol{q})^\top \dot{\boldsymbol{q}}
\end{aligned}$$

Describing angular velocity vector using quaternion

angular velocities:

$$\omega_\xi = 2 \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \end{bmatrix} \dot{\boldsymbol{q}}$$
$$\omega_\eta = 2 \begin{bmatrix} -q_2 & -q_3 & q_0 & q_1 \end{bmatrix} \dot{\boldsymbol{q}}$$
$$\omega_\zeta = 2 \begin{bmatrix} -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \dot{\boldsymbol{q}}$$

angular velocity vector

$$\boldsymbol{\omega} = 2H\dot{\boldsymbol{q}}$$

where

$$H \triangleq \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

Describing Lagrangian using quaternion

Lagrangian

$$L = T + \lambda_{\rm quat} Q_{\rm quat}$$

kinetic energy of a rotating rigid body:

$$T = \frac{1}{2}\boldsymbol{\omega}^\top J \boldsymbol{\omega} = \frac{1}{2}(2H\dot{\boldsymbol{q}})^\top J (2H\dot{\boldsymbol{q}}) = 2\dot{\boldsymbol{q}}^\top (H^\top J H)\dot{\boldsymbol{q}}$$

or

$$T = \frac{1}{2}\boldsymbol{\omega}^\top J \boldsymbol{\omega} = \frac{1}{2}(-2\dot{H}\boldsymbol{q})^\top J (-2\dot{H}\boldsymbol{q}) = 2\boldsymbol{q}^\top (\dot{H}^\top J \dot{H})\boldsymbol{q}$$

constraint on quaternion:

$$Q_{\rm quat} = \boldsymbol{q}^\top \boldsymbol{q} - 1$$

$\lambda_{\rm quat}$: Lagrange multiplier

Computing Lagrange equation

partial derivatives of T w.r.t. $\dot{\boldsymbol{q}}$ and $\boldsymbol{q}$:

$$\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = 4(H^\top J H)\dot{\boldsymbol{q}}, \qquad \frac{\partial T}{\partial \boldsymbol{q}} = 4(\dot{H}^\top J \dot{H})\boldsymbol{q}$$

since $\dot{H}\boldsymbol{q} = -H\dot{\boldsymbol{q}}$

$$\frac{\partial T}{\partial \boldsymbol{q}} = 4(\dot{H}^\top J)\dot{H}\boldsymbol{q} = 4(\dot{H}^\top J)(-H\dot{\boldsymbol{q}}) = -4(\dot{H}^\top J H)\dot{\boldsymbol{q}}$$

time derivative of $\partial T / \partial \dot{\boldsymbol{q}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = 4(H^\top J H)\ddot{\boldsymbol{q}} + 4(\dot{H}^\top J H + H^\top J \dot{H})\dot{\boldsymbol{q}}$$

since $\dot{H}\dot{\boldsymbol{q}} = \mathbf{0}$ yields $(H^\top J \dot{H})\dot{\boldsymbol{q}} = (H^\top J)\dot{H}\dot{\boldsymbol{q}} = \mathbf{0}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = 4(H^\top J H)\ddot{\boldsymbol{q}} + 4(\dot{H}^\top J H)\dot{\boldsymbol{q}}$$

Computing Lagrange equation

contribution of kinetic energy T to Lagrange equation:

$$\frac{\partial T}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = -4(\dot{H}^\top J H)\dot{\boldsymbol{q}} - \left\{ 4(H^\top J H)\ddot{\boldsymbol{q}} + 4(\dot{H}^\top J H)\dot{\boldsymbol{q}} \right\}$$
$$= -4(H^\top J H)\ddot{\boldsymbol{q}} - 8(\dot{H}^\top J H)\dot{\boldsymbol{q}}$$

contribution of constraint Q_{quat} to Lagrange equation:

$$\frac{\partial Q_{\mathrm{quat}}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial Q_{\mathrm{quat}}}{\partial \dot{\boldsymbol{q}}} = 2\boldsymbol{q}$$

Lagrange equation of motion

$$-4(H^\top J H)\ddot{\boldsymbol{q}} - 8(\dot{H}^\top J H)\dot{\boldsymbol{q}} + 2\lambda_{\mathrm{quat}}\boldsymbol{q} = \mathbf{0}_4$$

Dynamic equations describing spatial rotation

multiply H to Lagrange equation of motion:

$$-4H(H^\top JH)\ddot{\boldsymbol{q}} - 8H(\dot{H}^\top JH)\dot{\boldsymbol{q}} + 2\lambda_{\rm quat} H\boldsymbol{q} = \boldsymbol{0}_3$$

since $HH^\top = I_{3\times 3}$ and $H\boldsymbol{q} = \boldsymbol{0}$:

$$JH\ddot{\boldsymbol{q}} + 2(H\dot{H}^\top JH)\dot{\boldsymbol{q}} = \boldsymbol{0}_3$$

matrix J is regular:

$$H\ddot{\boldsymbol{q}} = -2J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}}$$

since $\dot{Q}_{\rm quat} = 2\boldsymbol{q}^\top\dot{\boldsymbol{q}}$ and $\ddot{Q}_{\rm quat} = 2\boldsymbol{q}^\top\ddot{\boldsymbol{q}} + 2\dot{\boldsymbol{q}}^\top\dot{\boldsymbol{q}}$, equation for stabilizing constraint $Q_{\rm quat} = 0$ is given by

$$-\boldsymbol{q}^\top\ddot{\boldsymbol{q}} = r(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

where

$$r(\boldsymbol{q}, \dot{\boldsymbol{q}}) \stackrel{\triangle}{=} \dot{\boldsymbol{q}}^\top\dot{\boldsymbol{q}} + 2\nu\boldsymbol{q}^\top\dot{\boldsymbol{q}} + \frac{1}{2}\nu^2(\boldsymbol{q}^\top\boldsymbol{q} - 1) \quad (\nu\text{: positive constant})$$

Dynamic equations describing spatial rotation

differential equations:

$$
\begin{aligned}
-\boldsymbol{q}^\top \ddot{\boldsymbol{q}} &= r(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\
H\ddot{\boldsymbol{q}} &= -2J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}}
\end{aligned}
$$

combining the two equations:

$$
\begin{aligned}
\begin{bmatrix} -\boldsymbol{q}^\top \\ H \end{bmatrix} \ddot{\boldsymbol{q}} &= \begin{bmatrix} r(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ -2J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}} \end{bmatrix} \\
\hat{H}\ddot{\boldsymbol{q}} &= \begin{bmatrix} r(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ -2J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}} \end{bmatrix}
\end{aligned}
$$

where

$$
\hat{H} \triangleq \begin{bmatrix} -\boldsymbol{q}^\top \\ H \end{bmatrix}
$$

Dynamic equations describing spatial rotation

$$\hat{H}\hat{H}^\top = \begin{bmatrix} -\boldsymbol{q}^\top \\ H \end{bmatrix} \begin{bmatrix} -\boldsymbol{q} & H^\top \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}^\top\boldsymbol{q} & -(H\boldsymbol{q})^\top \\ -H\boldsymbol{q} & HH^\top \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \mathbf{0}_3^\top \\ \mathbf{0}_3 & I_{3\times 3} \end{bmatrix} = I_{4\times 4}$$

matrix $\hat{H}$ is orthogonal:

$$\ddot{\boldsymbol{q}} = \hat{H}^\top \begin{bmatrix} r(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ -2J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}} \end{bmatrix}$$

$$= \begin{bmatrix} -\boldsymbol{q} & H^\top \end{bmatrix} \begin{bmatrix} r(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ -2J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}} \end{bmatrix}$$

$$= -r(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} - 2H^\top J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}}$$

Dynamic equations describing spatial rotation

$$2H\dot{H}^\top = 2\begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}\begin{bmatrix} -\dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 \\ \dot{q}_0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & \dot{q}_0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & \dot{q}_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} = \left[\ \boldsymbol{\omega}\times\ \right] = \left[\ (2H\dot{\boldsymbol{q}})\times\ \right]$$

matrix $H\dot{H}^\top$ represents outer product with $H\dot{\boldsymbol{q}}$

$$\ddot{\boldsymbol{q}} = -r(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} - 2H^\top J^{-1}(H\dot{H}^\top JH)\dot{\boldsymbol{q}}$$

$$\Downarrow$$

$$\ddot{\boldsymbol{q}} = -r(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} - 2H^\top J^{-1}\left\{(H\dot{\boldsymbol{q}}) \times (JH\dot{\boldsymbol{q}})\right\}$$

Dynamic equations describing spatial rotation

Equation of rotation

4 generalized coordinates:

$$\boldsymbol{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

4 differential equations:

$$\ddot{\boldsymbol{q}} = -r(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} - 2H^{\top} J^{-1} \{(H\dot{\boldsymbol{q}}) \times (JH\dot{\boldsymbol{q}})\}$$

Dynamic equations describing spatial rotation

Canonical form for numerical computation

$$\dot{\boldsymbol{q}} = \boldsymbol{p}$$
$$\dot{\boldsymbol{p}} = -r(\boldsymbol{q}, \boldsymbol{p})\boldsymbol{q} - 2H^\top J^{-1}\left\{(H\boldsymbol{p}) \times (JH\boldsymbol{p})\right\}$$

state variable vector

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}$$

Comparison summary

- a set of rotation matrix elements
  - 12 state variables (9 for orientation and 3 for angular velocity)
  - 12 equations (6 differential + 6 algebraic)
- quaternion
  - 4 generalized coordinates (8 state variables)
  - quadratic expressions, no trigonometric functions
  - no singularity, implying no gimbal lock or no instability
- a set of Euler angles
  - 3 generalized coordinates (6 state variables)
  - trigonometric functions
  - singularity, causing gimbal lock or instability

Lagrange equation of forced spatial rotation

$\boldsymbol{\tau}_{\rm quat} = [\,\tau_0,\ \tau_1,\ \tau_2,\ \tau_3\,]^\top$ a set of generalized torques corresponding to quaternion $\boldsymbol{q} = [\,q_0,\ q_1,\ q_2,\ q_3\,]^\top$

$$W = \boldsymbol{\tau}_{\rm quat}^\top \boldsymbol{q} = \tau_0 q_0 + \tau_1 q_1 + \tau_2 q_2 + \tau_3 q_3$$

contribution of work W to Lagrange equation:

$$\frac{\partial W}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial W}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{\tau}_{\rm quat}$$

Lagrange equation of motion:

$$-4(H^\top J H)\ddot{\boldsymbol{q}} - 8(\dot{H}^\top J H)\dot{\boldsymbol{q}} + 2\lambda_{\rm quat}\boldsymbol{q} + \boldsymbol{\tau}_{\rm quat} = \mathbf{0}_4$$

$$\ddot{\boldsymbol{q}} = -r(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} - 2H^\top J^{-1}\left\{(H\dot{\boldsymbol{q}}) \times (JH\dot{\boldsymbol{q}}) - \frac{1}{8}H\boldsymbol{\tau}_{\rm quat}\right\}$$

Lagrange equation of forced spatial rotation

Principle of virtual works

$$\boldsymbol{\omega} = 2H\dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{\tau}_{\rm quat} = (2H)^{\top}\boldsymbol{\tau}$$

since $H\boldsymbol{\tau}_{\rm quat} = 2HH^{\top}\boldsymbol{\tau} = 2\boldsymbol{\tau}$, equation of rotation turns into:

$$\ddot{\boldsymbol{q}} = -r(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{q} - 2H^{\top}J^{-1}\left\{(H\dot{\boldsymbol{q}}) \times (JH\dot{\boldsymbol{q}}) - \frac{1}{4}\boldsymbol{\tau}\right\}$$

Lagrange equation of forced spatial rotation

Canonical form for numerical computation

$$\dot{\boldsymbol{q}} = \boldsymbol{p}$$

$$\dot{\boldsymbol{p}} = -r(\boldsymbol{q}, \boldsymbol{p})\boldsymbol{q} - 2H^{\top}J^{-1}\left\{(H\boldsymbol{p}) \times (JH\boldsymbol{p}) - \frac{1}{4}\boldsymbol{\tau}\right\}$$

state variable vector

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix}$$

Sample Programs

- class **RigidBody**
- class **RigidBody_Cuboid**

class **RigidBody_Cuboid** is a subclass of class **RigidBody**

Sample Programs

file RigidBody.m

```
classdef RigidBody
    properties
        density;
        mass;
        inertia_matrix;
        inertia_matrix_inverse;
        rotation_matrix;
        omega;
        q;
        dotq;
        H;
    end
    methods
        function obj = RigidBody (m, J)
            obj.mass = m;
```

Sample Programs

file RigidBody_Cuboid.m

```
classdef RigidBody_Cuboid < RigidBody
    properties
        a, b, c;
    end
    methods
        function obj = RigidBody_Cuboid (rho, a, b, c)
            obj@RigidBody(1,eye(3));
            m = rho*a*b*c;
            Jx = (1/12)*m*(b^2+c^2);
            Jy = (1/12)*m*(c^2+a^2);
            Jz = (1/12)*m*(a^2+b^2);
            J = diag([Jx, Jy, Jz]);
            obj = obj.mass_and_inertia_matrix(m, J);
            obj.density = rho;
            obj.a = a;
```

Sample Programs

file RigidBody_Cuboid_test.m

```
body = RigidBody_Cuboid(1, 4, 4, 8);
clf;
body.draw;
xlim([-10,10]); ylim([-10,10]); zlim([-10,10]);
xlabel('x'); ylabel('y'); zlabel('z');
pbaspect([1 1 1]);
grid on;
view([-75, 30]);
```

Sample Programs

file rotation_quaternion.m

```
body = RigidBody_Cuboid(1, 4, 4, 8);
alpha = 1000; % positive large constant for CSM

ext = @(t) external_torque(t);
rotation_quaternion_ODE = @(t,s) rotation_quaternion_ODE_
tf = 20;
interval = [0,tf];
sinit = [1;0;0;0; 0;0;0;0];
[time, s] = ode45(rotation_quaternion_ODE, interval, sini
```

Sample Programs

file rotation_quaternion.m

```
function dots = rotation_quaternion_ODE_params (t,s, body
    q = s(1:4); dotq = s(5:8);
    ddotq = body.calculate_ddotq (q, dotq, alpha, ext(t))
    dots = [dotq;ddotq];
end

function tau = external_torque(t)
    if t <= 5
        tau = [12.00; 0.00; 0.00];
    elseif t <= 10
        tau = [ 0.00; -12.00; 0.00];
    else
        tau = [0;0;0];
    end
```

Summary

Planar rotation

- described by angle θ and angular velocity ω
- Lagrangian formulation yields equation of planar rotation

Spatial rotation

- described by rotation matrix R under geometric constraints
- Lagrangian approach derives Euler's equation of rotation
- equation of forced rotation

Quaternion

- a set of four variables under one constraint
- differential eq. w.r.t. quaternion describing spacial rotation

Quaternion

Report #5 due date : Dec. 9 (Mon.) 1:00 AM

(1) Show that $R(\boldsymbol{q})$ is orthogonal.

(2) Show $\dot{A}\boldsymbol{q} = A\dot{\boldsymbol{q}}$, $\dot{B}\boldsymbol{q} = B\dot{\boldsymbol{q}}$, and $\dot{C}\boldsymbol{q} = C\dot{\boldsymbol{q}}$.

(3) Show $H\boldsymbol{q} = \mathbf{0}$.

(4) Show $\dot{H}\dot{\boldsymbol{q}} = \mathbf{0}$.

(5) Show $H\dot{\boldsymbol{q}} = -\dot{H}\boldsymbol{q}$ and $\boldsymbol{\omega} = -2\dot{H}\boldsymbol{q}$.

(6) Show $HH^\top = I_{3\times 3}$.

(7) Show $H^\top H\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{q}} = (1/2)H^\top\boldsymbol{\omega}$.

Dynamic Simulation of Rotation

Report #6 due date : Dec. 16 (Mon.) 1:00 AM

Simulate the rotation of a rigid cylinder in 3D space. Define class **RigidBody_Cylinder** as a subclass of **RigidBody**. Use appropriate values of geometrical and physical parameters of the cylinder. Apply torque vectors with different directions.

Appendix: vector calculus

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ are three-dimensional vectors given as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Inner product between $\boldsymbol{x}$ and $\boldsymbol{y}$ is described as

$$\boldsymbol{x}^\top \boldsymbol{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Thus, partial derivatives of the inner product with respect to column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are given as follows:

$$\frac{\partial(\boldsymbol{x}^\top \boldsymbol{y})}{\partial \boldsymbol{x}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boldsymbol{y}, \quad \frac{\partial(\boldsymbol{x}^\top \boldsymbol{y})}{\partial \boldsymbol{y}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{x}$$

Since the inner product is a scalar, the above derivatives are three-dimensional column vectors.

Appendix: vector calculus

Outer product between $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \triangleq \left[\, \boldsymbol{x} \times \,\right] \boldsymbol{y}$$

Note that $\left[\, \boldsymbol{x} \times \,\right]$ is a 3×3 skew-symmetric matrix.

$$\left[\, \boldsymbol{x} \times \,\right] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Appendix: vector calculus

Partial derivatives of the outer product with respect to row vectors $\boldsymbol{x}^\top$ and $\boldsymbol{y}^\top$:

$$
\frac{\partial(\boldsymbol{x} \times \boldsymbol{y})}{\partial \boldsymbol{x}^\top} = \begin{bmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{bmatrix} = \left[\; -\boldsymbol{y}\times \; \right],
$$

$$
\frac{\partial(\boldsymbol{x} \times \boldsymbol{y})}{\partial \boldsymbol{y}^\top} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} = \left[\; \boldsymbol{x}\times \; \right]
$$

Since the outer product is a three-dimensional column vector, the above derivatives are 3×3 matrices.

Appendix: vector calculus

Let $\boldsymbol{x}$ be a three-dimensional vector and A be a 3×3 symmetric matrix independent of $\boldsymbol{x}$. Quadratic form is described as:

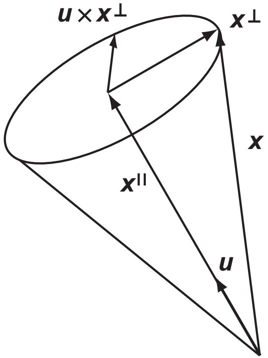
$$\begin{aligned} \boldsymbol{x}^\top A \boldsymbol{x} &= \left[\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \right] \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] \\ &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 \end{aligned}$$

Partial derivatie of the quadratic form with respect to $\boldsymbol{x}$ is:

$$\begin{aligned} \frac{\partial \boldsymbol{x}^\top A \boldsymbol{x}}{\partial \boldsymbol{x}} &= \left[\begin{array}{c} 2a_{11}x_1 + 2a_{12}x_2 + 2a_{13}x_3 \\ 2a_{22}x_2 + 2a_{12}x_1 + 2a_{23}x_3 \\ 2a_{33}x_3 + 2a_{13}x_3 + 2a_{23}x_3 \end{array} \right] \\ &= 2A\boldsymbol{x} \end{aligned}$$

Since the quadratic form is a scalar, the above derivative is a three-dimensional column vector.

Appendix: deriving quaternion description



$\boldsymbol{u} = [\, u_x,\ u_y,\ u_z\,]^\top$ unit vector

$R(\boldsymbol{u}, \alpha)$ rotation around $\boldsymbol{u}$ by angle α

$\boldsymbol{x}$ arbitrary vector

decompose $\boldsymbol{x}$ into two components

$$\boldsymbol{x} = \boldsymbol{x}^{\parallel} + \boldsymbol{x}^{\perp}$$

$$\boldsymbol{x}^{\parallel} \parallel \boldsymbol{u}, \qquad \boldsymbol{x}^{\perp} \perp \boldsymbol{u}$$

$\boldsymbol{x}^{\parallel}$ is the projection of $\boldsymbol{x}$ to a line specified by unit vector $\boldsymbol{u}$:

$$\boldsymbol{x}^{\parallel} = (\boldsymbol{u}^\top \boldsymbol{x})\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^\top)\boldsymbol{x},$$

$$\boldsymbol{x}^{\perp} = \boldsymbol{x} - \boldsymbol{x}^{\parallel} = (I_{3\times 3} - \boldsymbol{u}\boldsymbol{u}^\top)\boldsymbol{x}$$

vectors $\boldsymbol{u}$, $\boldsymbol{x}^{\perp}$, and $\boldsymbol{u} \times \boldsymbol{x}^{\perp}$ form a right-handed coordinate system

Appendix: deriving quaternion description

rotation $R(\boldsymbol{u}, \alpha)$ transforms $\boldsymbol{x}^{\parallel}$ into itself:

$$R\boldsymbol{x}^{\parallel} = \boldsymbol{x}^{\parallel}.$$

rotation $R(\boldsymbol{u}, \alpha)$ transforms $\boldsymbol{x}^{\perp}$ into $C_\alpha \boldsymbol{x}^{\perp} + S_\alpha \boldsymbol{u} \times \boldsymbol{x}^{\perp}$:

$$R\boldsymbol{x}^{\perp} = C_\alpha \boldsymbol{x}^{\perp} + S_\alpha \boldsymbol{u} \times \boldsymbol{x}^{\perp}$$

Thus

$$R\boldsymbol{x} = R(\boldsymbol{x}^{\parallel} + \boldsymbol{x}^{\perp}) = \boldsymbol{x}^{\parallel} + C_\alpha \boldsymbol{x}^{\perp} + S_\alpha \boldsymbol{u} \times \boldsymbol{x}^{\perp}.$$

since $\boldsymbol{u} \times \boldsymbol{x}^{\parallel} = \boldsymbol{0}$

$$\boldsymbol{u} \times \boldsymbol{x}^{\perp} = \boldsymbol{u} \times (\boldsymbol{x}^{\parallel} + \boldsymbol{x}^{\perp}) = \boldsymbol{u} \times \boldsymbol{x} = \left[\ \boldsymbol{u}\times\ \right] \boldsymbol{x}$$

$$\begin{aligned} R\boldsymbol{x} &= (\boldsymbol{u}\boldsymbol{u}^\top)\boldsymbol{x} + C_\alpha (I_{3\times 3} - \boldsymbol{u}\boldsymbol{u}^\top)\boldsymbol{x} + S_\alpha \left[\ \boldsymbol{u}\times\ \right] \boldsymbol{x} \\ &= \left\{ C_\alpha I_{3\times 3} + (1 - C_\alpha)\boldsymbol{u}\boldsymbol{u}^\top + S_\alpha \left[\ \boldsymbol{u}\times\ \right] \right\} \boldsymbol{x}. \end{aligned}$$

Appendix: deriving quaternion description

rotation around unit vector $\boldsymbol{u}$ by angle α:

$$R = C_\alpha I_{3\times 3} + (1 - C_\alpha)\boldsymbol{u}\boldsymbol{u}^\top + S_\alpha \left[\ \boldsymbol{u}\times\ \right]$$
$$= \begin{bmatrix} C_\alpha + \overline{C}_\alpha u_x^2 & \overline{C}_\alpha u_x u_y - S_\alpha u_z & \overline{C}_\alpha u_x u_z + S_\alpha u_y \\ \overline{C}_\alpha u_y u_x + S_\alpha u_z & C_\alpha + \overline{C}_\alpha u_y^2 & \overline{C}_\alpha u_y u_z - S_\alpha u_x \\ \overline{C}_\alpha u_z u_x - S_\alpha u_y & \overline{C}_\alpha u_z u_y + S_\alpha u_x & C_\alpha + \overline{C}_\alpha u_z^2 \end{bmatrix}$$

where $\overline{C}_\alpha = 1 - C_\alpha$
Define $q_0 = \cos(\alpha/2)$:

$$C_\alpha = 2q_0^2 - 1, \qquad \overline{C}_\alpha = 2\sin^2\frac{\alpha}{2}, \qquad S_\alpha = 2q_0 \sin\frac{\alpha}{2}$$

Define $[\, q_1,\ q_2,\ q_3 \,]^\top = \sin(\alpha/2)[\, u_x,\ u_y,\ u_z \,]^\top$:

$$R = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix}$$

Appendix: algebraic description of quaternion

R : a sequence of rotations P followed by Q
P, Q, R : denoted by quaternions
$\boldsymbol{p} = [\, p_0,\ p_1,\ p_2,\ p_3 \,]^\top$, $\boldsymbol{q} = [\, q_0,\ q_1,\ q_2,\ q_3 \,]^\top$, $\boldsymbol{r} = [\, r_0,\ r_1,\ r_2,\ r_3 \,]^\top$

$$R(\boldsymbol{r}) = R(\boldsymbol{p})R(\boldsymbol{q}),$$

$$\Updownarrow$$

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2 \\ p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3 \\ p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1 \end{bmatrix} \tag{1}$$

Appendix: algebraic description of quaternion

Define numbers of which units are given by 1, i, j, and k.
Four units satisfy:

$$
\begin{aligned}
&i^2 = j^2 = k^2 = -1, \\
&ij = k, \quad jk = i, \quad ki = j, \\
&ji = -k, \quad kj = -i, \quad ik = -j
\end{aligned}
$$

Multiplication among i, j, and k circulates but does not commute.
Numbers p, q, r:

$$
\begin{aligned}
p &= p_0 + p_1 i + p_2 j + p_3 k, \\
q &= q_0 + q_1 i + q_2 j + q_3 k, \\
r &= r_0 + r_1 i + r_2 j + r_3 k
\end{aligned}
$$

Appendix: algebraic description of quaternion

Product pq:

$$
\begin{aligned}
r = pq &= (p_0 + p_1 i + p_2 j + p_3 k)(q_0 + q_1 i + q_2 j + q_3 k) \\
&= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) \\
&+ (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2) i \\
&+ (p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3) j \\
&+ (p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1) k
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3 \\ p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2 \\ p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3 \\ p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1 \end{bmatrix} \tag{2}
$$

(1) and (2) are equivalent each other.

Appendix: Euler angles

a set of 3-2-3 Euler angles:

$$
\begin{aligned}
R(\phi, \theta, \psi) &= R_3(\phi) R_2(\theta) R_3(\psi) \\
&= \begin{bmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{bmatrix} \begin{bmatrix} C_\theta & & S_\theta \\ & 1 & \\ -S_\theta & & C_\theta \end{bmatrix} \begin{bmatrix} C_\psi & -S_\psi & \\ S_\psi & C_\psi & \\ & & 1 \end{bmatrix}
\end{aligned}
$$

Rotation matrix

$$
\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & \\ 1/\sqrt{2} & 1/\sqrt{2} & \\ & & 1 \end{bmatrix}
$$

corresponds to an infinite number of sets of Euler angles satisfying $\theta = 0$ and $\phi + \psi = \pi/4$. This singularity causes

- gimbal lock
- instability in solving equation of rotation

Appendix: Euler angles

a set of 1-2-3 Euler angles:

$$R(\phi,\theta,\psi) = R_1(\phi)R_2(\theta)R_3(\psi)$$
$$= \begin{bmatrix} 1 & & \\ & C_\phi & -S_\phi \\ & S_\phi & C_\phi \end{bmatrix} \begin{bmatrix} C_\theta & & S_\theta \\ & 1 & \\ -S_\theta & & C_\theta \end{bmatrix} \begin{bmatrix} C_\psi & -S_\psi & \\ S_\psi & C_\psi & \\ & & 1 \end{bmatrix}$$

Rotation matrix

$$\begin{bmatrix} & & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & \\ -1/\sqrt{2} & 1/\sqrt{2} & \end{bmatrix}$$

corresponds to an infinite number of sets of Euler angles satisfying $\theta = \pi/2$ and $\phi + \psi = \pi/4$.

Any set of Euler angles has singularity.

Appendix: Euler angles

computing angular velocity vector for a set of 3-2-3 Euler angles:

$$R^\top = R_3^\top(\psi) R_2^\top(\theta) R_3^\top(\phi)$$

$$\dot{R} = \dot{R}_3(\phi) R_2(\theta) R_3(\psi) + R_3(\phi) \dot{R}_2(\theta) R_3(\psi) + R_3(\phi) R_2(\theta) \dot{R}_3(\psi)$$

$$\begin{aligned} \left[\ \boldsymbol{\omega}\times\ \right] &= R^\top \dot{R} \\ &= R_3^\top(\psi) R_2^\top(\theta) R_3^\top(\phi) \dot{R}_3(\phi) R_2(\theta) R_3(\psi) \\ &+ R_3^\top(\psi) R_2^\top(\theta) \dot{R}_2(\theta) R_3(\psi) \\ &+ R_3^\top(\psi) \dot{R}_3(\psi) \end{aligned}$$

angular velocity vector:

$$\begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix} = \begin{bmatrix} -S_\theta C_\psi \\ S_\theta S_\psi \\ C_\theta \end{bmatrix} \dot{\phi} + \begin{bmatrix} S_\psi \\ C_\psi \\ 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\psi}$$