Analytical Mechanics: Variational Principles

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Agenda

Statics

Variation principle in statics

$$\text{minimize} \quad I = U - W$$

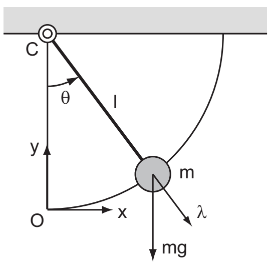
under constraint

$$\text{minimize} \quad I = U - W$$
$$\text{subject to} \quad R = 0$$

Solutions

- analytically solve $\delta I = 0$
- numerical optimization (`fminbnd` or `fmincon`)

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C
τ: external torque around C, θ: angle around C
Given τ, derive θ at equilibrium.

Statics in variational form

U	potential energy
W	work done by external forces/torques

Variational principle in statics

Internal energy $I = U - W$ reaches to its minimum at equilibrium:

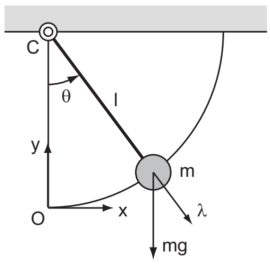
$$I = U - W \rightarrow \text{minimum}$$

Statics in variational form

Solutions:

1. Solve
$$\text{minimize} \quad I = U - W$$
analytically
2. Solve
$$\text{minimize} \quad I = U - W$$
numerically
3. Solve
$$\delta I = 0$$
analytically

Example (simple pendulum)



$$U = mgl(1 - \cos\theta), \quad W = \tau\theta$$
$$I = mgl(1 - \cos\theta) - \tau\theta$$

Example (simple pendulum)

Solve

$$\text{minimize} \quad I = mgl(1 - \cos\theta) - \tau\theta$$

analytically

$$\Downarrow$$

$$\frac{\partial I}{\partial \theta} = mgl \sin\theta - \tau = 0$$

Equilibrium of moment around C

Example (simple pendulum)

Solve

$$\text{minimize} \quad I = mgl(1-\cos\theta) - \tau\theta \qquad (-\pi \le \theta \le \pi)$$

numerically

$$\Downarrow$$

Apply `fminbnd` to minimize a function numerically

Example (simple pendulum)

Sample Programs

- minimizing internal energy
- internal energy of simple pendulm

Example (simple pendulum)

Result

```
>> internal_energy_simple_pendulum_min

thetamin =

    0.5354


Imin =

   -0.0261
```


Example (simple pendulum)

Solve

$$\delta I = 0$$

analytically

$$\Downarrow$$

$$I = mgl(1-\cos\theta) - \tau\theta$$
$$I + \delta I = mgl(1-\cos(\theta+\delta\theta)) - \tau(\theta+\delta\theta)$$

Example (simple pendulum)

Note that $\cos(\theta+\delta\theta) = \cos\theta - (\sin\theta)\delta\theta$:

$$I = mgl(1-\cos\theta) - \tau\theta$$
$$I + \delta I = mgl(1-\cos\theta + (\sin\theta)\delta\theta) - \tau(\theta+\delta\theta)$$

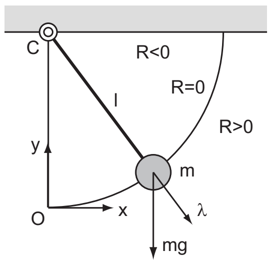
$$\Downarrow$$

$$\delta I = mgl(\sin\theta)\delta\theta - \tau\delta\theta$$
$$= (mgl\sin\theta - \tau)\delta\theta \equiv 0, \qquad \forall\delta\theta$$

$$\Downarrow$$

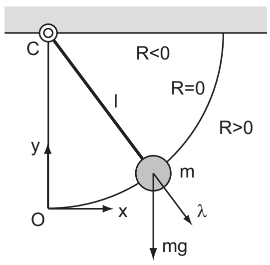
$$mgl\sin\theta - \tau = 0$$

Example (pendulum in Cartesian coordinates)



simple pendulum of length l and mass m suspended at point C
$[x, y]^\top$: position of mass
$[f_x, f_y]^\top$: external force applied to mass
Given $[f_x, f_y]^\top$, derive $[x, y]^\top$ at equilibrium.

Example (pendulum in Cartesian coordinates)



geometric constraint
distance between C and mass $= l$

$$R \triangleq \{x^2 + (y-l)^2\}^{1/2} - l = 0$$

Statics under single constraint

U	potential energy
W	work done by external forces/torques
R	geometric constraint

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraint $R = 0$:

$$\text{minimize} \quad U - W$$
$$\text{subject to} \quad R = 0$$

Statics under single constraint

Solutions:

1. Solve

$$\begin{aligned} \text{minimize} \quad & U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

analytically

2. Solve

$$\begin{aligned} \text{minimize} \quad & U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

numerically

Statics under single constraint

Solve

$$\begin{aligned} \text{minimize} \quad & U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

analytically

$$\Downarrow$$

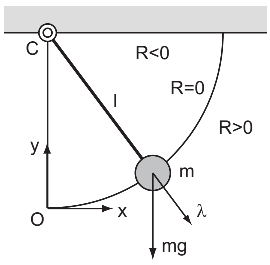
minimize $I = U - W - \lambda R$

λ: Lagrange's multiplier

$$\Downarrow$$

$$\delta I = \delta(U - W - \lambda R) = 0$$

Example (pendulum in Cartesian coordinates)



$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \left\{x^2 + (y-l)^2\right\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

$$I = mgy - (f_x x + f_y y) - \lambda \left[\left\{x^2 + (y-l)^2\right\}^{1/2} - l\right]$$

Note that $\delta R = R_x\, \delta x + R_y\, \delta y$, where

$$\begin{aligned} R_x &\triangleq \frac{\partial R}{\partial x} = x\left\{x^2 + (y-l)^2\right\}^{-1/2} \\ R_y &\triangleq \frac{\partial R}{\partial y} = (y-l)\left\{x^2 + (y-l)^2\right\}^{-1/2} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \delta I &= mg\,\delta y - f_x \delta x - f_y \delta y - \lambda R_x \delta x - \lambda R_y \delta y \\ &= (-f_x - \lambda R_x)\delta x + (mg - f_y - \lambda R_y)\delta y \equiv 0, \qquad \forall \delta x, \delta y \end{aligned}$$

Example (pendulum in Cartesian coordinates)

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ -mg \end{bmatrix}$ grav. force, $\begin{bmatrix} f_x \\ f_y \end{bmatrix}$ ext. force, $\lambda \underline{\begin{bmatrix} R_x \\ R_y \end{bmatrix}}$ constraint force

gradient vector ($\perp$ to $R = 0$)

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \\ R &= 0 \end{aligned}$$

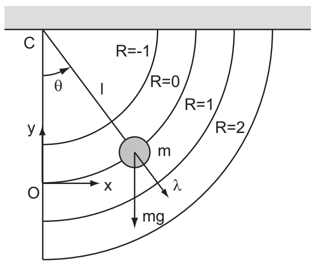
$$\Downarrow$$

we can determine position of mass $[\,x,\, y\,]^\top$ and magnitude of constraint force λ

Example (pendulum in Cartesian coordinates)

Note

$$\begin{aligned} I &= U - W - \lambda R \\ &= U - (W + \lambda R) \end{aligned}$$



λ magnitude of a constraint force

R distance along the force

constraint force $\perp$ contour $R =$ constant

λR work done by a constraint force

$W + \lambda R$ work done by external & constraint forces

Statics under single constraint

Solve

$$\begin{aligned} \text{minimize} \quad & I = U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

numerically

$$\Downarrow$$

Apply `fmincon` to minimize a function numerically under constraints

Note: "Optimization Toolbox" is needed to use `fmincon`

Example (pendulum in Cartesian coordinates)

Sample Programs

- minimizing internal energy (Cartesian)
- internal energy of simple pendulm (Cartesian)
- constraints

Example (pendulum in Cartesian coordinates)

Result:

```
>> internal_energy_pendulum_Cartesian_min
Local minimum found that satisfies the constraints.

<stopping criteria details>

qmin =
    1.4001
    3.4281

Imin =
   -0.4897
```


Statics under multiple constraints

U	potential energy
W	work done by external forces/torques
R_1, R_2	geometric constraints

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraints $R_1 = 0$ and $R_2 = 0$:

$$\text{minimize} \quad U - W$$
$$\text{subject to} \quad R_1 = 0, \quad R_2 = 0$$

$$\delta I = \delta(U - W - \lambda_1 R_1 - \lambda_2 R_2) = 0$$

λ_1, λ_2: Lagrange's multipliers

Dynamics

Lagrangian

$$\mathcal{L} = T - U + W$$
$$\mathcal{L} = T - U + W + \lambda R \qquad \text{(under constraint)}$$

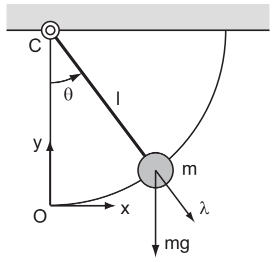
Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{0}$$

Solutions

- numerical ODE solver (ode45)
- constraint stabilization method (CSM)

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C
τ: external torque around C at time t, θ: angle around C at time t
Derive the motion of the pendulum.

Dynamics in variational form

T	kinetic energy
U	potential energy
W	work done by external forces/torques

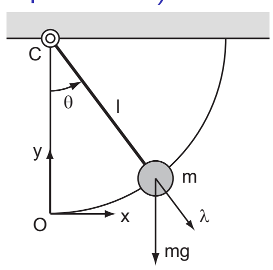
Lagrangian

$$\mathcal{L} = T - U + W$$

Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

Example (simple pendulum)



$$T = \frac{1}{2}(ml^2)\dot{\theta}^2$$
$$U = mgl(1 - \cos\theta), \qquad W = \tau\theta$$

Example (simple pendulum)

Lagrangian

$$\mathcal{L} = \frac{1}{2}(ml^2)\dot{\theta}^2 - mgl(1 - \cos\theta) + \tau\theta$$

partial derivatives

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta + \tau, \qquad \frac{\partial L}{\partial \dot{\theta}} = (ml^2)\dot{\theta}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

Lagrange equation of motion

$$-mgl\sin\theta + \tau - ml^2\ddot{\theta} = 0$$

Example (simple pendulum)

Equation of the pendulum motion

$$ml^2\ddot{\theta} = -mgl\sin\theta + \tau$$

$$\Downarrow$$

Canonical form of ordinary differential equation

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \frac{1}{ml^2}(\tau - mgl\sin\theta)$$

can be solved numerically by an ODE solver

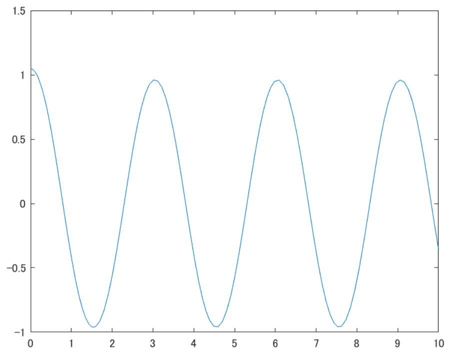
Example (simple pendulum)

Sample Programs

- solve the equation of motion of simple pendulum
- equation of motion of simple pendulum
- external torque

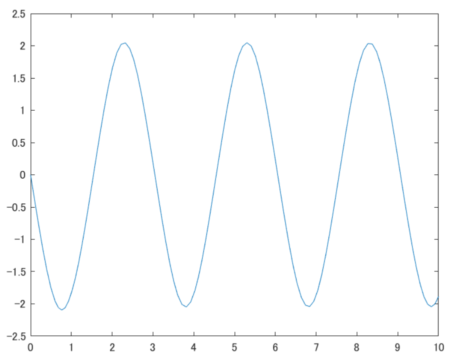
Example (simple pendulum)

Result



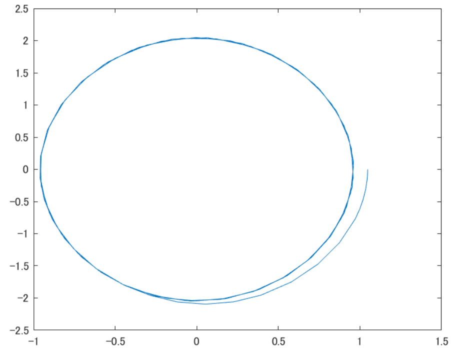
Example (simple pendulum)

Result



Example (simple pendulum)

Result



Example (pendulum with viscous friction)

Assumptions

viscous friction around supporting point C works
viscous friction causes a negative torque around C
magnitude of the torque is proportional to angular velocity

$$\text{viscous friction torque} = -b\dot{\theta} \quad (b\text{: positive constant})$$

Replacing τ by $\tau - b\dot{\theta}$:

$$(ml^2)\ddot{\theta} = (\tau - b\dot{\theta}) - mgl\sin\theta$$

$$\Downarrow$$

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \frac{1}{ml^2}(\tau - b\omega - mgl\sin\theta)$$

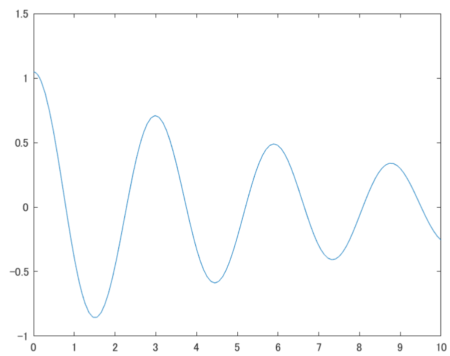
Example (pendulum with viscous friction)

Sample Programs

- solve the equation of motion of damped pendulum
- equation of motion of damped pendulum
- external torque

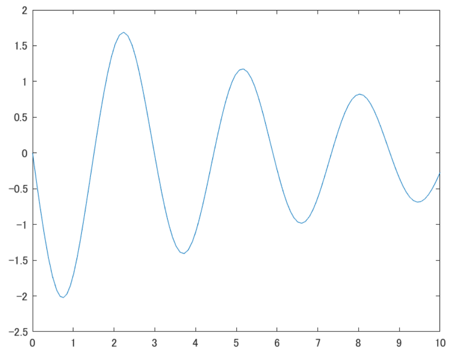
Example (pendulum with viscous friction)

Result



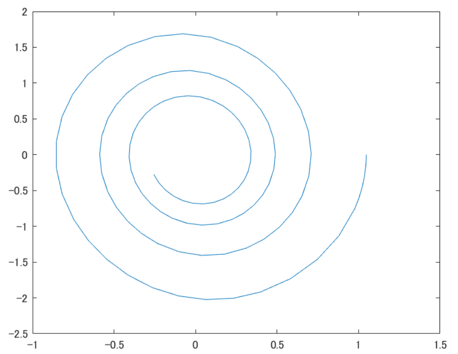
Example (pendulum with viscous friction)

Result

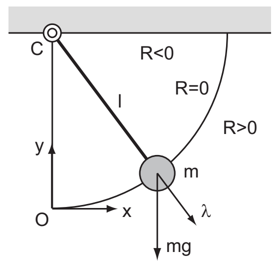


Example (pendulum with viscous friction)

Result



Example (pendulum in Cartesian coordinates)



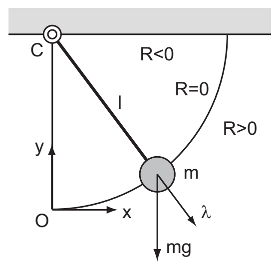
simple pendulum of length l and mass m suspended at point C

$[x, y]^{\top}$: position of mass at time t

$[f_x, f_y]^{\top}$: external force applied to mass at time t

Derive the motion of the pendulum in Cartesian coordinates.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass $= l$

$$R \triangleq \left\{x^2 + (y-l)^2\right\}^{1/2} - l = 0$$

Dynamics under single constraint

T	kinetic energy
U	potential energy
W	work done by external forces/torques

Lagrangian

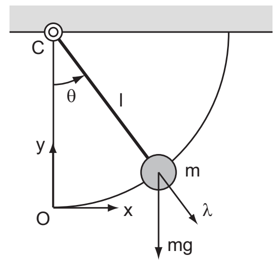
$$\mathcal{L} = T - U + W + \lambda R$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

Example (pendulum in Cartesian coordinates)



$$T = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\}$$

$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \left\{x^2 + (y-l)^2\right\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

Lagrangian

$$\mathcal{L} = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\} - mgy + f_x x + f_y y + \lambda\left[\left\{x^2 + (y-l)^2\right\}^{1/2} - l\right]$$

partial derivatives

$$\frac{\partial \mathcal{L}}{\partial x} = f_x + \lambda R_x, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -mg + f_y + \lambda R_y, \quad \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}$$

Lagrange equations of motion

$$f_x + \lambda R_x - m\ddot{x} = 0$$

$$-mg + f_y + \lambda R_y - m\ddot{y} = 0$$

Example (pendulum in Cartesian coordinates)

Lagrange equations of motion

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} + \left\{ -m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

gravitational external constraint inertial

dynamic equilibrium among forces

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$m\ddot{x} = f_x + \lambda R_x$$
$$m\ddot{y} = -mg + f_y + \lambda R_y$$
$$R = 0$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$m\ddot{x} = f_x + \lambda R_x$$
$$m\ddot{y} = -mg + f_y + \lambda R_y$$
$$R = 0$$

Mixture of differential and algebraic equations

$$\Downarrow$$

Difficult to solve the mixture of differential and algebraic equations

Constraint stabilization method (CSM)

Constraint stabilization

convert algebraic eq. to its almost equivalent differential eq.

algebraic eq. $\quad R = 0$

$$\Downarrow$$

differential eq. $\quad \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$

(α: large positive constant)

critical damping (converges to zero most quickly)

Constraint stabilization method (CSM)

Dynamic equation of motion under geometric constraint:

differential eq. $\quad \dfrac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{0}$

algebraic eq. $\quad R = 0$

$$\Downarrow$$

differential eq. $\quad \dfrac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{0}$

differential eq. $\quad \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$

can be solved numerically by an ODE solver.

Computing equation for constraint stabilization

Assume R depends on x and y: $R(x, y) = 0$

Differentiating $R(x, y)$ w.r.t time t:

$$\dot{R} = \frac{\partial R}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_x\dot{x} + R_y\dot{y}$$

Differentiating $R_x(x, y)$ and $R_y(x, y)$ w.r.t time t:

$$\dot{R}_x = \frac{\partial R_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R_x}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_{xx}\dot{x} + R_{xy}\dot{y}$$
$$\dot{R}_y = \frac{\partial R_y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R_y}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_{yx}\dot{x} + R_{yy}\dot{y}$$

Second order time derivative:

$$\begin{aligned}\ddot{R} &= (\dot{R}_x\dot{x} + R_x\ddot{x}) + (\dot{R}_y\dot{y} + R_y\ddot{y}) \\ &= (R_{xx}\dot{x} + R_{xy}\dot{y})\dot{x} + R_x\ddot{x} + (R_{yx}\dot{x} + R_{yy}\dot{y})\dot{y} + R_y\ddot{y}\end{aligned}$$

Computing equation for constraint stabilization

Second order time derivative:

$$\ddot{R} = \begin{bmatrix} R_x & R_y \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}\begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Equation to stabilize constraint:

$$-\begin{bmatrix} R_x & R_y \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix}\begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + 2\alpha(R_x\dot{x} + R_y\dot{y}) + \alpha^2 R$$

$$\Downarrow \quad v_x \triangleq \dot{x},\ v_y \triangleq \dot{y}$$

$$-\begin{bmatrix} R_x & R_y \end{bmatrix}\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x & v_y \end{bmatrix}\begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

Example (pendulum in Cartesian coordinates)

Equation for stabilizing constraint $R(x, y) = 0$:

$$-R_x\dot{v}_x - R_y\dot{v}_y = C(x, y, v_x, v_y)$$

where

$$C(x, y, v_x, v_y) = \begin{bmatrix} v_x & v_y \end{bmatrix}\begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

In this example

$$P = \{x^2 + (y-l)^2\}^{-1/2}, \quad R_x = xP, \quad R_y = (y-l)P$$
$$R_{xx} = P - x^2P^3, \quad R_{yy} = P - (y-l)^2P^3$$
$$R_{xy} = R_{yx} = -x(y-1)P^3$$

Example (pendulum in Cartesian coordinates)

Combining equations of motion and equation for constraint stabilization:

$$\dot{x} = v_x$$
$$\dot{y} = v_y$$
$$\begin{bmatrix} m & & -R_x \\ & m & -R_y \\ -R_x & -R_y & \end{bmatrix}\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \lambda \end{bmatrix} = \begin{bmatrix} f_x \\ -mg + f_y \\ C(x, y, v_x, v_y) \end{bmatrix}$$

five equations w.r.t. five unknown variables x, y, v_x, v_y and λ

given x, y, v_x, $v_y \Longrightarrow \dot{x}$, $\dot{y}$, $\dot{v}_x$, $\dot{v}_y$

This canonical ODE can be solved numerically by an ODE solver.

Example (pendulum in Cartesian coordinates)

Let $\boldsymbol{x} = [x,\, y]^\top$. Introducing gradient vector

$$\boldsymbol{g} = \begin{bmatrix} R_x \\ R_y \end{bmatrix}$$

yields

$$\dot{R} = \boldsymbol{g}^\top \dot{\boldsymbol{x}}$$

Introducing Hessian matrix

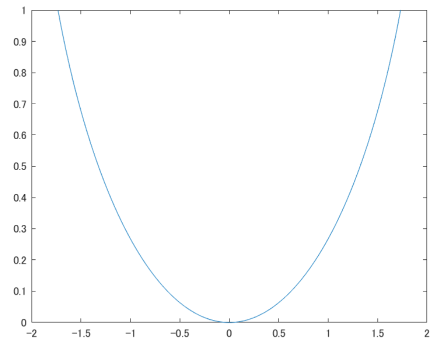
$$H = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}$$

yields

$$\ddot{R} = \boldsymbol{g}^\top \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^\top H \dot{\boldsymbol{x}}$$

Example (pendulum in Cartesian coordinates)

x–y



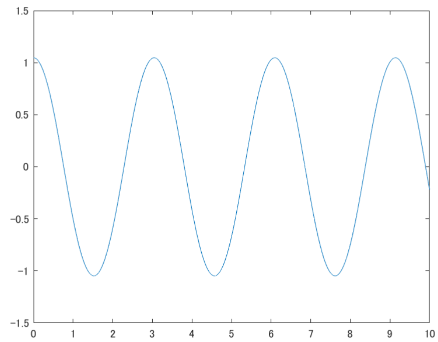
Example (pendulum in Cartesian coordinates)

Sample Programs

- solve the equation of motion of simple pendulum (Cartesian)
- equation of motion of simple pendulum (Cartesian)

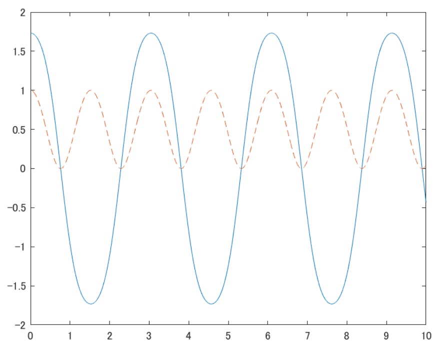
Example (pendulum in Cartesian coordinates)

t–computed θ



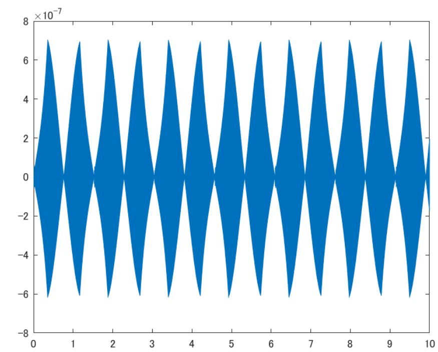
Example (pendulum in Cartesian coordinates)

t–x, y



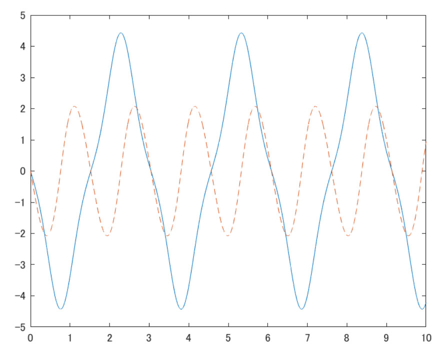
Example (pendulum in Cartesian coordinates)

t–constraint R



Example (pendulum in Cartesian coordinates)

t–v_x, v_y



Notice

Lagrangian

$$\begin{aligned} \mathcal{L} &= T - U + W + \lambda R \\ &= T - (U - W - \lambda R) \\ &= T - I \end{aligned}$$

Lagrangian is equal to the difference between kinetic energy and internal energy under a constraint

Summary

Variational principles

- statics $I = U - W$
- statics under constraint $I = U - W - \lambda R$

$$\delta I \equiv 0$$

- dynamics $\mathcal{L} = T - U + W$
- dynamics under constraint $\mathcal{L} = T - U + W + \lambda R$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{0}$$

- constraint stabilization method

Summary

How to solve a static problem

Solve (nonlinear) equations originated from variation
or
Numerically minimize internal energy

How to solve a dynamic problem

Step 1 Derive Lagrange equations of motion analytically
Step 2 Solve the derived equations numerically

Report

Report # 1 due date : Oct. 28 (Mon) 1:00 AM

Simulate the dynamic motion of a pendulum under viscous friction described with Cartesian coordinates x and y. Apply constraint stabilization method to convert the constraint into its almost equivalent ODE, then apply any ODE solver to solve a set of ODEs (equations of motion and equation for constraint stabilization) numerically.

Submit your report in pdf format to manaba+R

File name shoud be:
student number (11 digits) your name (without space).pdf
For example 12345678901HiraiShinichi.pdf

Report

Report # 2 due date : Nov. 4 (Mon) 1:00 AM

Assume that a system is described by four coordinates q_1 through q_4. Two constraints R_1 and R_2 are imposed on the system. Let $\boldsymbol{q} = [\,q_1,\ q_2,\ q_3,\ q_4\,]^\top$ and $\boldsymbol{R} = [\,R_1,\ R_2\,]^\top$. Let $\boldsymbol{g}_1$ and H_1 be gradient vector and Hessian matrix related to R_1 while $\boldsymbol{g}_2$ and H_2 be gradient vector and Hessian matrix related to R_2. Let J be Jacobian given by

$$J = \begin{bmatrix} \partial R_1/\partial q_1 & \partial R_1/\partial q_2 & \partial R_1/\partial q_3 & \partial R_1/\partial q_4 \\ \partial R_2/\partial q_1 & \partial R_2/\partial q_2 & \partial R_2/\partial q_3 & \partial R_2/\partial q_4 \end{bmatrix}$$

Show the following equations:

$$\dot{\boldsymbol{R}} = J\dot{\boldsymbol{q}}$$

$$\ddot{\boldsymbol{R}} = J\ddot{\boldsymbol{q}} + \begin{bmatrix} \dot{\boldsymbol{q}}^\top H_1 \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^\top H_2 \dot{\boldsymbol{q}} \end{bmatrix}$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$y = x^2$$

Let us change variable x to $x + \delta x$, then variable y changes to $y + \delta y$ accordingly.

$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x\,\delta x + (\delta x)^2 \\ &= x^2 + 2x\,\delta x \end{aligned}$$

Thus

$$\delta y = 2x\,\delta x$$

Appendix: Variational calculus

Small virtual deviation of variables or functions.

$$I = \int_0^T \{x(t)\}^2 \,\mathrm{d}t$$

Let us change function $x(t)$ to $x(t) + \delta x(t)$, then variable I changes to $I + \delta I$ accordingly.

$$\begin{aligned} I + \delta I &= \int_0^T \{x(t) + \delta x(t)\}^2 \,\mathrm{d}t \\ &= \int_0^T \{x(t)\}^2 + 2x(t)\,\delta x(t)\,\mathrm{d}t \end{aligned}$$

Thus

$$\delta I = \int_0^T 2x(t)\,\delta x(t)\,\mathrm{d}t$$

Appendix: Variational calculus

Variational operator δ

$\delta\theta$ virtual deviation of variable θ
$\delta f(\theta)$ virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\theta \to \theta + \delta\theta$

increment of function $f(\theta) \to f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$
$f(\theta) \to f(\theta) + \delta f(\theta)$

simple examples

$$\delta(5x) = 5\,\delta x \qquad \delta x^2 = 2x\,\delta x$$
$$\delta \sin\theta = (\cos\theta)\,\delta\theta, \qquad \delta\cos\theta = (-\sin\theta)\,\delta\theta$$

Appendix: Variational calculus

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$f(\theta) \to f(\theta) + \delta f(\theta)$

simple examples

$$\delta(5x) = 5\,\delta x \qquad \delta x^2 = 2x\,\delta x$$
$$\delta \sin\theta = (\cos\theta)\,\delta\theta, \qquad \delta\cos\theta = (-\sin\theta)\,\delta\theta$$

Appendix: Variational calculus

assume that θ depends on time t
virtual increment of function $\theta(t) \to \theta(t) + \delta\theta(t)$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \to \frac{\mathrm{d}}{\mathrm{d}t}(\theta + \delta\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$$
$$\int \theta \, \mathrm{d}t \to \int (\theta + \delta\theta) \, \mathrm{d}t = \int \theta \, \mathrm{d}t + \int \delta\theta \, \mathrm{d}t$$

variation of derivative and integral

$$\delta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$$
$$\delta \int \theta \, \mathrm{d}t = \int \delta\theta \, \mathrm{d}t$$

variational operator and differential/integral operator can commute

Appendix: Lagrange multiplier method

converts minimization (maximization) under conditions into minimization (maximization) without conditions.

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}) \\ &\text{subject to } g(\boldsymbol{x}) = 0 \end{aligned}$$

$$\Downarrow$$

$$\text{minimize } I(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \lambda g(\boldsymbol{x})$$

$$\Downarrow$$

$$\frac{\partial I}{\partial \boldsymbol{x}} = \frac{\partial f}{\partial \boldsymbol{x}} + \lambda \frac{\partial g}{\partial \boldsymbol{x}} = \boldsymbol{0}$$
$$\frac{\partial I}{\partial \lambda} = g(\boldsymbol{x}) = 0$$

Appendix: Lagrange multiplier method (example)

Length of each edge of a cube is given by x, y, and z.
Determine x, y, and z that minimizes the surface of the cube when the cube volume is constantly specified by a^3:

$$\begin{aligned} &\text{minimize } S(x, y, z) = 2xy + 2yz + 2zx \\ &\text{subject to } R(x, y, z) \triangleq xyz - a^3 = 0 \end{aligned}$$

Introducing Lagrange multiplier λ, the above conditional minimization can be converted into the following unconditional minimization:

$$\begin{aligned} \text{minimize } I(x, y, z, \lambda) &= S(x, y, z) + \lambda R(x, y, z) \\ &= 2xy + 2yz + 2zx + \lambda(xyz - a^3) \end{aligned}$$

Appendix: Lagrange multiplier method (example)

Calculating partial derivatives:

$$\frac{\partial I}{\partial x} = 2y + 2z - \lambda yz = 0 \quad (1)$$
$$\frac{\partial I}{\partial y} = 2z + 2x - \lambda zx = 0 \quad (2)$$
$$\frac{\partial I}{\partial z} = 2x + 2y - \lambda xy = 0 \quad (3)$$
$$\frac{\partial I}{\partial \lambda} = xyz - a^3 = 0 \quad (4)$$

Calculating $(1) \cdot x - (2) \cdot y$, we have

$$z(x - y) = 0,$$

which directly yields $x = y$. Similarly, we have $y = z$ and $z = x$.
Consequently, we concludes $x = y = z = a$.

Appendix: ODE solver

Let us solve van del Pol equation:

$$\ddot{x} - 2(1 - x^2)\dot{x} + x = 0$$

Canonical form:

$$\dot{x} = v$$
$$\dot{v} = 2(1 - x^2)\dot{x} - x$$

State variable vector:

$$\boldsymbol{q} = \begin{bmatrix} x \\ v \end{bmatrix}$$

Appendix: ODE solver (MATLAB)

File `van_der_Pol.m` describes the canonical form:

```
function dotq = van_der_Pol (t,q)
    x = q(1);
    v = q(2);
    dotx = v;
    dotv = 2*(1-x^2)*v - x;
    dotq = [dotx; dotv];
end
```

File name `van_der_Pol` should conincide with function name `van_der_Pol`.

Appendix: ODE solver (MATLAB)

File `van_der_Pol_solve.m` solves van der Pol equation numerically:

```
timestep=0.00:0.10:10.00;
qinit=[2.00;0.00];
[time,q]=ode45(@van_der_Pol,timestep,qinit);

% line style   solid -   broken -.   chain --   dotted :
plot(time,q(:,1),'-', time,q(:,2),'-.');
```

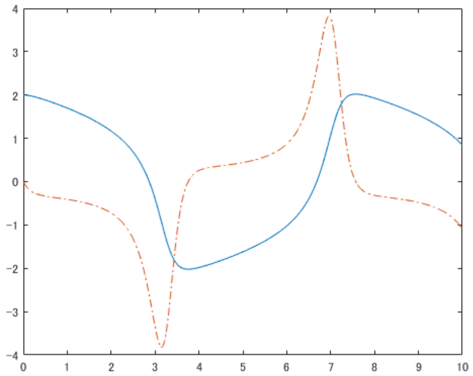

Appendix: ODE solver (MATLAB)

```
>> time
time =
         0
    0.1000
    0.2000
    0.3000
    0.4000

>> q
q =
    2.0000         0
    1.9917   -0.1504
    1.9721   -0.2338
    1.9461   -0.2822
    1.9163   -0.3125
```

The first and second columns corresponds to x and v.

Appendix: ODE solver (MATLAB)



Dynamics in variational form

Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ w.r.t.
a set of generalized coordinates $\boldsymbol{q}$ and its time derivative $\dot{\boldsymbol{q}}$
time integral of Lagrangian:

$$\text{action integral} = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t$$

Variational principle in dynamics

variation of action integral vanishes
for any geometrically admissible variation of $\boldsymbol{q}$

$$\text{V.I.} \triangleq \int_{t_1}^{t_2} \delta\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t \quad \equiv \quad 0$$

for any $\delta\boldsymbol{q}$ satisfying $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$

Dynamics in variational form

Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ w.r.t.
a set of generalized coordinates $\boldsymbol{q}$ and its time derivative $\dot{\boldsymbol{q}}$
time integral of Lagrangian:

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Variational principle in dynamics

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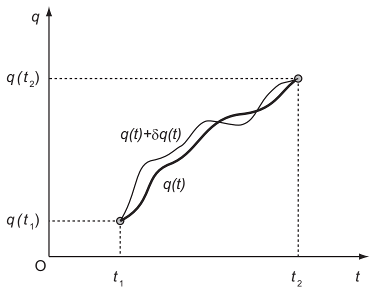
for any $\delta\boldsymbol{q}$ satisfying $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$

Dynamics in variational form



Lagrangian corresponding to $\boldsymbol{q} + \delta\boldsymbol{q}$:

$$\mathcal{L}(\boldsymbol{q} + \delta\boldsymbol{q}, \dot{\boldsymbol{q}} + \delta\dot{\boldsymbol{q}}, t) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^{\top} \delta\boldsymbol{q} + \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\dot{\boldsymbol{q}}$$

Dynamics in variational form

variation of $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$:

$$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^{\top} \delta\boldsymbol{q} + \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\dot{\boldsymbol{q}}$$

time integral of the second term:

$$\begin{aligned}
\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\dot{\boldsymbol{q}}\,\mathrm{d}t &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{q}\,\mathrm{d}t \\
&= \underbrace{\left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\boldsymbol{q}\right]_{t=t_1}^{t=t_2}}_{0 \text{ since } \delta\boldsymbol{q}(t_1) = 0 \text{ and } \delta\boldsymbol{q}(t_2) = 0} - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\boldsymbol{q}\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\boldsymbol{q}\,\mathrm{d}t
\end{aligned}$$

Dynamics in variational form

Variation of action integral:

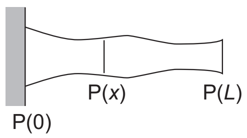
$$\begin{aligned}
\text{V.I.} &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^{\top} \delta\boldsymbol{q}\,\mathrm{d}t + \int_{t_1}^{t_2} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\boldsymbol{q}\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^{\top} \delta\boldsymbol{q}\,\mathrm{d}t \quad \equiv \quad 0 \qquad \forall\delta\boldsymbol{q}
\end{aligned}$$

$$\Downarrow$$

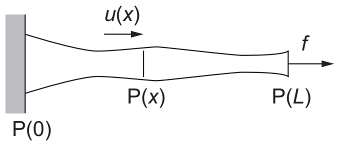
Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{0}$$

Example (extension of beam)



natural shape at time 0



deformed shape at time t

Deformation at time t is described by function $u(x, t)$ $(0 \le x \le L)$

Example (extension of beam)

E: Young's modulus at point $\mathrm{P}(x)$
A: Cross-sectional area at point $\mathrm{P}(x)$
ρ: density at point $\mathrm{P}(x)$

Assumption: $A(x)$, $E(x)$, and $\rho(x)$ do not change despite of extension, *i.e.* axial deformation is negligible.

Kinetic energy, elastic potential energy, and work done by external force

$$T = \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$

$$U = \int_0^L \frac{1}{2}EA \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$

$$W = f\,u(L, t)$$

Example (extension of beam)

Lagrangian

$$\mathcal{L} = T - U + W$$
$$= \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x - \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + f\, u(L,t)$$

Variation of Lagrangian

$$\delta\mathcal{L} = \int_0^L \rho A \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\delta u\,\mathrm{d}x - \int_0^L EA\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\delta u\,\mathrm{d}x + f\,\delta u(L,t)$$

Recall

$$\delta U = \int_0^L EA\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\delta u\,\mathrm{d}x$$
$$= EA\frac{\partial u}{\partial x}\delta u\Big|_{x=L} - \int_0^L \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)\delta u\,\mathrm{d}x$$

Example (extension of beam)

Time-integral of the first term of $\delta\mathcal{L}$:

$$\int_{t_1}^{t_2}\int_0^L \rho A\frac{\partial u}{\partial t}\frac{\partial}{\partial t}\delta u\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^L\int_{t_1}^{t_2} \rho A\frac{\partial u}{\partial t}\frac{\partial}{\partial t}\delta u\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^L \left\{\left[\rho A\frac{\partial u}{\partial t}\delta u\right]_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2}\frac{\partial}{\partial t}\left(\rho A\frac{\partial u}{\partial t}\right)\delta u\,\mathrm{d}t\right\}\mathrm{d}x$$
$$= \int_0^L\left\{-\int_{t_1}^{t_2}\rho A\frac{\partial^2 u}{\partial t^2}\delta u\,\mathrm{d}t\right\}\mathrm{d}x$$
$$= \int_{t_1}^{t_2}\int_0^L\left\{-\rho A\frac{\partial^2 u}{\partial t^2}\delta u\right\}\mathrm{d}x\,\mathrm{d}t$$

Example (extension of beam)

$$\mathrm{V.I.} = \int_{t_1}^{t_2}\delta\mathcal{L}\,\mathrm{d}t$$
$$= \int_{t_1}^{t_2}\int_0^L\left\{-\rho A\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)\right\}\delta u\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{t_1}^{t_2}\left\{-EA\frac{\partial u}{\partial x}\Big|_{x=L} + f\right\}\delta u(L,t)\,\mathrm{d}t$$

should be equal to 0 for any $\delta u(x,t)$

$$\Downarrow$$

$$-\rho A\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = 0, \qquad -EA\frac{\partial u}{\partial x}\Big|_{x=L} + f = 0$$

Example (extension of beam)

Equation of deformation (partial differential equation)

$$\rho A\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)$$

Boundary conditions

$$u(0,t) = 0$$
$$E(L,t)A(L,t)\frac{\mathrm{d}u}{\mathrm{d}x}(L,t) = f(t)$$

Initial conditions (example)

$$u(x,0) = 0, \quad \forall x \in [0, L]$$
$$\frac{\mathrm{d}u}{\mathrm{d}t}(x,0) = 0, \quad \forall x \in [0, L]$$

Example (extension of beam)

Assume that E. A, and ρ are constant:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $c = \sqrt{E/\rho}$

Given function $f(x)$, let

$$u(x,t) = f(x - ct)$$

Then

$$\partial u/\partial x = f'(x-ct), \quad \partial^2 u/\partial x^2 = f''(x-ct),$$
$$\partial u/\partial t = f'(x-ct)(-c), \quad \partial^2 u/\partial t^2 = f''(x-ct)(-c)^2$$

Thus, $f(x - ct)$ is one solution of the PDE.

Example (extension of beam)

Given function $g(x)$, let

$$u(x,t) = g(x + ct)$$

Then

$$\partial u/\partial x = g'(x+ct), \quad \partial^2 u/\partial x^2 = g''(x+ct),$$
$$\partial u/\partial t = g'(x+ct)(+c), \quad \partial^2 u/\partial t^2 = g''(x+ct)(+c)^2$$

Thus, $g(x + ct)$ is one solution of the PDE.

Rayleigh wave

Solution $f(x - ct)$: wave propagating at speed $+c$
Solution $g(x + ct)$: wave propagating at speed $-c$