Finite Element Modeling

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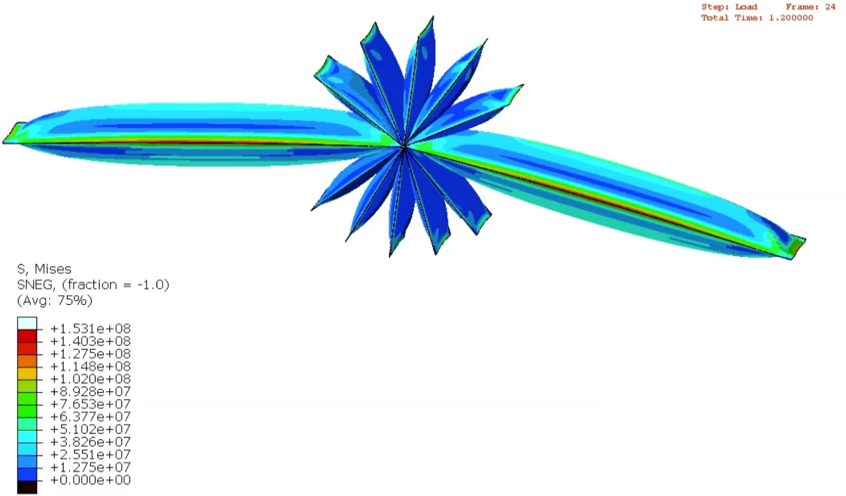
Agenda

1. Two/Three-dimensional Finite Element Method

Finite Element Method (FEM)

inflatable link simulation



Finite Element Method (FEM)

Integral forms (weak forms) apply

strain potential energy (one-dim. beam)

$$U = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x$$

kinetic energy (one-dim. beam)

$$T = \int_0^L \frac{1}{2} \rho A \left(\frac{\partial u}{\partial t} \right)^2 \mathrm{d}x$$

How calculate energies in integral forms?

Finite Element Method (FEM)

divide

$$\int_0^L = \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \int_{x_3}^{x_4} + \int_{x_4}^{x_5}$$

apply piecewise linear approximation

$$\int_{x_i}^{x_j} = \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

synthsize

$$\int_0^L = \frac{1}{2} \begin{bmatrix} u_1 & u_2 & \cdots & u_5 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{bmatrix}$$

Two/Three-dimensional Deformation

one-dimensional deformation

extensional strain ε

Young's modulus E

strain potential energy density $\quad \dfrac{1}{2}E\varepsilon^2$

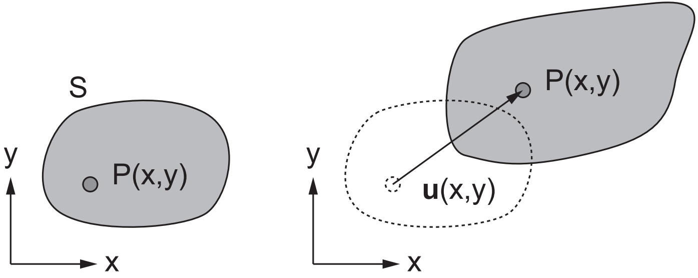
two/three-dimensional deformation

extensional & shear strains $\rightarrow$ strain vector $\boldsymbol{\varepsilon}$

Lamé's constants λ, μ $\rightarrow$ elasticity matrix $\lambda I_\lambda + \mu I_\mu$

strain potential energy density $\quad \dfrac{1}{2}\boldsymbol{\varepsilon}^\top(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}$

Two-dimensional Deformation

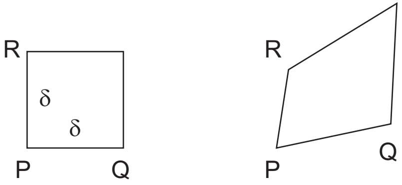


natural state moved and deformed state

displacement vector

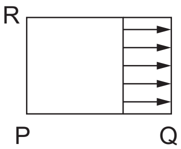
$$\boldsymbol{u}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$$

Two-dimensional Deformation

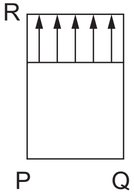


natural   deformed and rotated

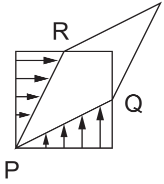
Two-dimensional Deformation



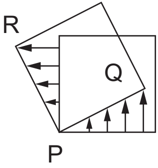
extension along x-axis



extension along y-axis



shear deformation



rotational motion

Two-dimensional Deformation

$$\frac{\partial u}{\partial x} = \text{extension along } x\text{-axis} \quad \frac{\partial v}{\partial y} = \text{extension along } y\text{-axis}$$

$$\frac{\partial v}{\partial x} = \text{shear} + \text{rotation} \quad \frac{\partial u}{\partial y} = \text{shear} - \text{rotation}$$

$$\Downarrow$$

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Two-dimensional Deformation

strain vector

$$\boldsymbol{\varepsilon} \triangleq \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix}$$

Two-dimensional Deformation

Strain potential energy density

linear isotropic elastic material

$$\frac{1}{2}\boldsymbol{\varepsilon}^{\top}(\lambda I_{\lambda} + \mu I_{\mu})\boldsymbol{\varepsilon}$$

where λ and μ are Lamé's constants and

$$I_{\lambda} = \begin{bmatrix} 1 & 1 & \\ 1 & 1 & \\ & & \end{bmatrix}, \quad I_{\mu} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

Two-dimensional Deformation

Volume element

$$h \, \mathrm{d}S = h \, \mathrm{d}x \, \mathrm{d}y$$

Strain potential energy

$$U = \int_S \frac{1}{2} \boldsymbol{\varepsilon}^\top (\lambda I_\lambda + \mu I_\mu) \boldsymbol{\varepsilon} \, h \, \mathrm{d}S$$

Two-dimensional Deformation

Volume element

$$h\,\mathrm{d}S = h\,\mathrm{d}x\,\mathrm{d}y$$

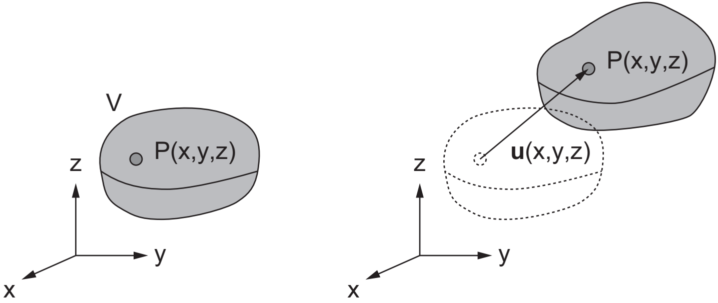
Strain potential energy

$$U = \int_S \frac{1}{2}\boldsymbol{\varepsilon}^\top (\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}\; h\,\mathrm{d}S$$

Kinetic energy

$$T = \int_S \frac{1}{2}\rho\,\dot{\boldsymbol{u}}^\top \dot{\boldsymbol{u}}\; h\,\mathrm{d}S$$

Three-dimensional Deformation

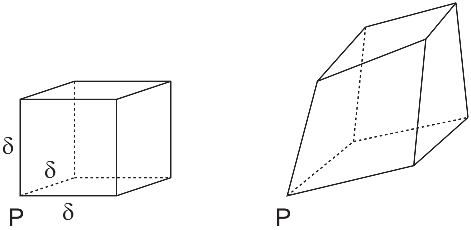


natural state　　moved and deformed state

displacement vector

$$\boldsymbol{u}(x, y, z) = \begin{bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{bmatrix}$$

Three-dimensional Deformation



natural deformed and rotated

Three-dimensional Deformation

	u	v	w
$\partial/\partial x$	ext. along x	shr + rot in xy	shr − rot in zx
$\partial/\partial y$	shr − rot in xy	ext. along y	shr + rot in yz
$\partial/\partial z$	shr + rot in zx	shr − rot in yz	ext. along z

$$2 \cdot \text{shear in } yz\text{-plane} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$2 \cdot \text{shear in } zx\text{-plane} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2 \cdot \text{shear in } xy\text{-plane} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Three-dimensional Deformation

Cauchy strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$2\varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Three-dimensional Deformation

strain vector

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{bmatrix}$$

Three-dimensional Deformation

Strain potential energy density

linear isotropic elastic material

$$\frac{1}{2}\boldsymbol{\varepsilon}^{\top}(\lambda I_{\lambda} + \mu I_{\mu})\boldsymbol{\varepsilon}$$

$$I_{\lambda} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ \hline & & & & & \\ & & & & & \\ & & & & & \end{array}\right], \quad I_{\mu} = \left[\begin{array}{ccc|ccc} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right]$$

Three-dimensional Deformation

Volume element

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Strain potential energy

$$U = \int_V \frac{1}{2}\boldsymbol{\varepsilon}^\top(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}\,\mathrm{d}V$$

Three-dimensional Deformation

Volume element

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

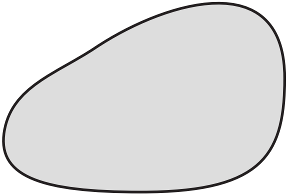
Strain potential energy

$$U = \int_V \frac{1}{2}\boldsymbol{\varepsilon}^\top(\lambda I_\lambda + \mu I_\mu)\boldsymbol{\varepsilon}\,\mathrm{d}V$$

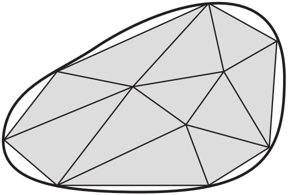
Kinetic energy

$$T = \int_V \frac{1}{2}\rho\,\dot{\boldsymbol{u}}^\top\dot{\boldsymbol{u}}\,\mathrm{d}V$$

Two-dimensional FEM

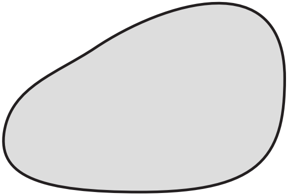


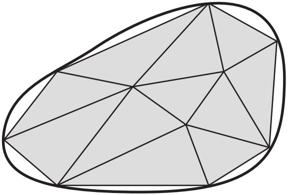
region S



cover by triangles

Two-dimensional FEM





$$\int_S \mathrm{d}S \approx \sum_{\text{triangles}} \int_{\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k} \mathrm{d}S$$

Two-dimensional FEM

assume density ρ and thickness h are constants
kinetic energy of $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$

$$T_{i,j,k} = \int_{\triangle} \frac{1}{2} \rho \, \dot{\boldsymbol{u}}^{\top} \dot{\boldsymbol{u}} \, h \, \mathrm{d}S$$

$$= \frac{1}{2} \left[\begin{array}{ccc} \dot{\boldsymbol{u}}_i^{\top} & \dot{\boldsymbol{u}}_j^{\top} & \dot{\boldsymbol{u}}_k^{\top} \end{array} \right] \frac{\rho h \triangle}{12} \left[\begin{array}{ccc} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{array} \right] \left[\begin{array}{c} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \end{array} \right]$$

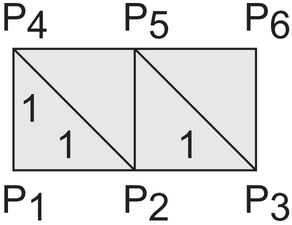
(see `Chapter_2_Finite_Element_Approximation.pdf` at `www.ritsumei.ac.jp/~hirai/edu/common/soft_robotics/`)

Two-dimensional FEM

Partial inertia matrix

$$M_{i,j,k} = \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}$$

Example (inertia matrix)



assume $\rho h \triangle / 12$ is constantly equal to 1
partial inertia matrices

$$M_{1,2,4} = M_{2,3,5} = M_{5,4,2} = M_{6,5,3} = \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}.$$

Example (inertia matrix)

total kinetic energy

$$T = \frac{1}{2} \dot{\boldsymbol{u}}_{\mathrm{N}}^{\top} \, M \, \dot{\boldsymbol{u}}_{\mathrm{N}}$$

$$= \frac{1}{2} \left[\begin{array}{cccc} \dot{\boldsymbol{u}}_1^{\top} & \dot{\boldsymbol{u}}_2^{\top} & \cdots & \dot{\boldsymbol{u}}_6^{\top} \end{array} \right] \left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{array} \right] \left[\begin{array}{c} \dot{\boldsymbol{u}}_1 \\ \dot{\boldsymbol{u}}_2 \\ \vdots \\ \dot{\boldsymbol{u}}_6 \end{array} \right]$$

M: inertia matrix (6×6 block matrix)

Example (inertia matrix)

$$M_{1,2,4} = \left[\begin{array}{c|c|c} (1,1)\text{ block} & (1,2)\text{ block} & (1,4)\text{ block} \\ \hline (2,1)\text{ block} & (2,2)\text{ block} & (2,4)\text{ block} \\ \hline (4,1)\text{ block} & (4,2)\text{ block} & (4,4)\text{ block} \end{array}\right]$$

contribution of $M_{1,2,4}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\ \hline I_{2\times 2} & 2I_{2\times 2} & & I_{2\times 2} & & \\ \hline & & & & & \\ \hline I_{2\times 2} & I_{2\times 2} & & 2I_{2\times 2} & & \\ \hline & & & & & \\ \hline & & & & & \end{array}\right]$$

Example (inertia matrix)

$$M_{2,3,5} = \left[\begin{array}{c|c|c} (2,2) \text{ block} & (2,3) \text{ block} & (2,5) \text{ block} \\ \hline (3,2) \text{ block} & (3,3) \text{ block} & (3,5) \text{ block} \\ \hline (5,2) \text{ block} & (5,3) \text{ block} & (5,5) \text{ block} \end{array}\right]$$

contribution of $M_{2,3,5}$ to M

$$\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & \\ \hline & I_{2\times 2} & 2I_{2\times 2} & & I_{2\times 2} & \\ \hline & & & & & \\ \hline & I_{2\times 2} & I_{2\times 2} & & 2I_{2\times 2} & \\ \hline & & & & & \end{array}\right]$$

Example (inertia matrix)

$$
M_{5,4,2} = \left[\begin{array}{c|c|c} (5,5)\ \text{block} & (5,4)\ \text{block} & (5,2)\ \text{block} \\ \hline (4,5)\ \text{block} & (4,4)\ \text{block} & (4,2)\ \text{block} \\ \hline (2,5)\ \text{block} & (2,4)\ \text{block} & (2,2)\ \text{block} \end{array} \right]
$$

contribution of $M_{5,4,2}$ to M

$$
\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & 2I_{2\times 2} & & I_{2\times 2} & I_{2\times 2} & \\ \hline & & & & & \\ \hline & I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} & \\ \hline & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} & \\ \hline & & & & & \end{array} \right]
$$

Example (inertia matrix)

$$
M_{6,5,3} = \left[\begin{array}{c|c|c} (6,6) \text{ block} & (6,5) \text{ block} & (6,3) \text{ block} \\ \hline (5,6) \text{ block} & (5,5) \text{ block} & (5,3) \text{ block} \\ \hline (3,6) \text{ block} & (3,5) \text{ block} & (3,3) \text{ block} \end{array} \right]
$$

contribution of $M_{6,5,3}$ to M

$$
\left[\begin{array}{c|c|c|c|c|c}
 & & & & & \\ \hline
 & & & & & \\ \hline
 & & 2I_{2\times 2} & & I_{2\times 2} & I_{2\times 2} \\ \hline
 & & & & & \\ \hline
 & & I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} \\ \hline
 & & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2}
\end{array} \right]
$$

Example (inertia matrix)

inertia matrix

$$
\begin{aligned}
M &= M_{1,2,4} \oplus M_{2,3,5} \oplus M_{5,4,2} \oplus M_{6,5,3} \\
&= \begin{bmatrix}
2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\
I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} & 2I_{2\times 2} & \\
 & I_{2\times 2} & 4I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} \\
I_{2\times 2} & 2I_{2\times 2} & & 4I_{2\times 2} & I_{2\times 2} & \\
 & 2I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} \\
 & & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2}
\end{bmatrix}
\end{aligned}
$$

Two-dimensional FEM

assume λ, μ and h are constants
strain potential energy stored in $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$

$$U_{i,j,k} = \int_{\triangle} \frac{1}{2}\, \boldsymbol{\varepsilon}^{\top} (\lambda I_{\lambda} + \mu I_{\mu}) \boldsymbol{\varepsilon}\, h \, \mathrm{d}S$$

$$= \frac{1}{2} \left[\begin{array}{ccc} \boldsymbol{u}_i^{\top} & \boldsymbol{u}_j^{\top} & \boldsymbol{u}_k^{\top} \end{array} \right] K_{i,j,k} \left[\begin{array}{c} \boldsymbol{u}_i \\ \boldsymbol{u}_j \\ \boldsymbol{u}_k \end{array} \right]$$

where

$$K_{i,j,k} = \lambda J_{\lambda}^{i,j,k} + \mu J_{\mu}^{i,j,k}$$

(see `Chapter_2_Finite_Element_Approximation.pdf`)

Two-dimensional FEM

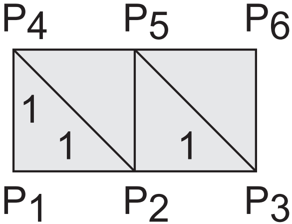
$$\boldsymbol{a} = \frac{1}{2\triangle} \begin{bmatrix} y_j - y_k \\ y_k - y_i \\ y_i - y_j \end{bmatrix}, \qquad \boldsymbol{b} = \frac{-1}{2\triangle} \begin{bmatrix} x_j - x_k \\ x_k - x_i \\ x_i - x_j \end{bmatrix}$$

$$H_\lambda = \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^\top & \boldsymbol{a}\boldsymbol{b}^\top \\ \boldsymbol{b}\boldsymbol{a}^\top & \boldsymbol{b}\boldsymbol{b}^\top \end{bmatrix} h\triangle$$

$$H_\mu = \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top & \boldsymbol{b}\boldsymbol{a}^\top \\ \boldsymbol{a}\boldsymbol{b}^\top & 2\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{a}\boldsymbol{a}^\top \end{bmatrix} h\triangle$$

$1, 4, 2, 5, 3, 6$ rows and columns of H_λ, H_μ $\rightarrow$
$1, 2, 3, 4, 5, 6$ rows and columns of $J_\lambda^{i,j,k}$, $J_\mu^{i,j,k}$

Example (stiffness matrix)



assume $h = 2$

stiffness matrix

$$K = K_{1,2,4} \oplus K_{2,3,5} \oplus K_{5,4,2} \oplus K_{6,5,3}$$

Example (stiffness matrix)

assume λ and μ are constants over region

$$\begin{aligned} K &= K_{1,2,4} \oplus K_{2,3,5} \oplus K_{5,4,2} \oplus K_{6,5,3} \\ &= (\lambda J_\lambda^{1,2,4} + \mu J_\mu^{1,2,4}) \oplus (\lambda J_\lambda^{2,3,5} + \mu J_\mu^{2,3,5}) \oplus \cdots \\ &= \lambda (J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus \cdots) + \mu (J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus \cdots) \\ &= \lambda J_\lambda + \mu J_\mu \end{aligned}$$

where

$$\begin{aligned} J_\lambda &= J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus J_\lambda^{5,4,2} \oplus J_\lambda^{6,5,3} \\ J_\mu &= J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus J_\mu^{5,4,2} \oplus J_\mu^{6,5,3} \end{aligned}$$

Example (stiffness matrix)

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\boldsymbol{a} = [\,-1,\ 1,\ 0\,]^\top$ and $\boldsymbol{b} = [\,-1,\ 0,\ 1\,]^\top$

$$H_\lambda = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$J_\lambda^{1,2,4} = \left[\begin{array}{cc|cc|cc} 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & -1 \\ \hline -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Example (stiffness matrix)

$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$: $\quad \boldsymbol{a} = [\,-1,\ 1,\ 0\,]^\top$ and $\boldsymbol{b} = [\,-1,\ 0,\ 1\,]^\top$

$$H_\mu = \left[\begin{array}{ccc|ccc} 3 & -2 & -1 & 1 & -1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ \hline 1 & 0 & -1 & 3 & -1 & -2 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{array}\right]$$

$$J_\mu^{1,2,4} = \left[\begin{array}{rr|rr|rr} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ \hline -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ \hline -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{array}\right]$$

Example (stiffness matrix)

$$J_{\lambda}^{1,2,4} = J_{\lambda}^{2,3,5} = J_{\lambda}^{5,4,2} = J_{\lambda}^{6,5,3}$$

$$J_{\mu}^{1,2,4} = J_{\mu}^{2,3,5} = J_{\mu}^{5,4,2} = J_{\mu}^{6,5,3}$$

Example (stiffness matrix)

contribution of $J_{\lambda}^{1,2,4}$ to J_{λ}

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
\hline
-1 & -1 & 1 & 0 & & & 0 & 1 & & & & \\
0 & 0 & 0 & 0 & & & 0 & 0 & & & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & & & 0 & 0 & & & & \\
-1 & -1 & 1 & 0 & & & 0 & 1 & & & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}\right]
$$

Example (stiffness matrix)

contribution of $J_\lambda^{2,3,5}$ to J_λ

$$
\left[
\begin{array}{cc|cc|cc|cc|cc|cc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & 1 & 1 & -1 & 0 & & & 0 & -1 & & \\
 & & 1 & 1 & -1 & 0 & & & 0 & -1 & & \\
\hline
 & & -1 & -1 & 1 & 0 & & & 0 & 1 & & \\
 & & 0 & 0 & 0 & 0 & & & 0 & 0 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & 0 & 0 & 0 & 0 & & & 0 & 0 & & \\
 & & -1 & -1 & 1 & 0 & & & 0 & 1 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}
\right]
$$

Example (stiffness matrix)

contribution of $J_{\lambda}^{5,4,2}$ to J_{λ}

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & 0 & 0 & & & 0 & 0 & 0 & 0 & & \\
 & & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
 & & 0 & 0 & & & 0 & 0 & 0 & 0 & & \\
\hline
 & & 0 & -1 & & & -1 & 0 & 1 & 1 & & \\
 & & 0 & -1 & & & -1 & 0 & 1 & 1 & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}\right]
$$

Example (stiffness matrix)

contribution of $J_{\lambda}^{6,5,3}$ to J_{λ}

$$
\left[\begin{array}{cc|cc|cc|cc|cc|cc}
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & 0 & 0 & & & 0 & 0 & 0 & 0 \\
 & & & & 0 & 1 & & & 1 & 0 & -1 & -1 \\
\hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & 0 & 1 & & & 1 & 0 & -1 & -1 \\
 & & & & 0 & 0 & & & 0 & 0 & 0 & 0 \\
\hline
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1 \\
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1
\end{array}\right]
$$

Example (stiffness matrix)

$$J_\lambda = J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus J_\lambda^{5,4,2} \oplus J_\lambda^{6,5,3}$$

$$= \left[\begin{array}{cc|cc|cc|cc|cc|cc}
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
\hline
-1 & -1 & 2 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & & \\
0 & 0 & 1 & 2 & -1 & 0 & 1 & 0 & -1 & -2 & & \\
\hline
 & & -1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 \\
 & & 0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 \\
\hline
0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
-1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 & & \\
\hline
 & & 0 & -1 & 0 & 1 & -1 & 0 & 2 & 1 & -1 & -1 \\
 & & -1 & -2 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & 0 \\
\hline
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1 \\
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1
\end{array}\right]$$

Example (stiffness matrix)

$$J_\mu = J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus J_\mu^{5,4,2} \oplus J_\mu^{6,5,3}$$

$$= \left[\begin{array}{rr|rr|rr|rr|rr|rr}
3 & 1 & -2 & -1 & & & -1 & 0 & & & & \\
1 & 3 & 0 & -1 & & & -1 & -2 & & & & \\ \hline
-2 & 0 & 6 & 1 & -2 & -1 & 0 & 1 & -2 & -1 & & \\
-1 & -1 & 1 & 6 & 0 & -1 & 1 & 0 & -1 & -4 & & \\ \hline
 & & -2 & 0 & 3 & 0 & & & 0 & 1 & -1 & -1 \\
 & & -1 & -1 & 0 & 3 & & & 1 & 0 & 0 & -2 \\ \hline
-1 & -1 & 0 & 1 & & & 3 & 0 & -2 & 0 & & \\
0 & -2 & 1 & 0 & & & 0 & 3 & -1 & -1 & & \\ \hline
 & & -2 & -1 & 0 & 1 & -2 & -1 & 6 & 1 & -2 & 0 \\
 & & -1 & -4 & 1 & 0 & 0 & -1 & 1 & 6 & -1 & -1 \\ \hline
 & & & & -1 & 0 & & & -2 & -1 & 3 & 1 \\
 & & & & -1 & -2 & & & 0 & -1 & 1 & 3
\end{array}\right]$$

Example (stiffness matrix)

stiffness matrix

$$K = \lambda J_{\lambda} + \mu J_{\mu}$$

$\lambda,\ \mu$	material-specific
$J_{\lambda},\ J_{\mu}$	geometric

strain potential energy

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}^{\top} K \, \boldsymbol{u}_{\mathrm{N}}$$

Lagrange equation

Kinetic and strain potential energies

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}^{\top} M \, \dot{\boldsymbol{u}}_{\mathrm{N}}, \qquad U = \frac{1}{2}\boldsymbol{u}_{\mathrm{N}}^{\top} K \, \boldsymbol{u}_{\mathrm{N}}$$

Work done by external forces

$$W = \boldsymbol{f}^{\top}\boldsymbol{u}_{\mathrm{N}}$$

Constraints

$$\boldsymbol{R} = A^{\top}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}(t) = \mathbf{0}$$

Lagrangian

$$\mathcal{L} = T - U + W + \boldsymbol{\lambda}^{\top}\boldsymbol{R}$$

Lagrange equation

Lagrange equation of motion and deformation

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}} = \mathbf{0}$$

$$\Downarrow$$

$$-K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} + A\boldsymbol{\lambda} - M\ddot{\boldsymbol{u}}_{\mathrm{N}} = \mathbf{0}$$

$$\Downarrow$$

$$\dot{\boldsymbol{u}}_{\mathrm{N}} = \boldsymbol{v}_{\mathrm{N}}$$

$$M\dot{\boldsymbol{v}}_{\mathrm{N}} - A\boldsymbol{\lambda} = -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f}$$

Lagrange equation

Equation for constraint stabilization

$$\ddot{\boldsymbol{R}} + 2\alpha\dot{\boldsymbol{R}} + \alpha^2\boldsymbol{R} = \boldsymbol{0}$$

$$\Downarrow$$

$$(A^\top\ddot{\boldsymbol{u}}_\mathrm{N} - \ddot{\boldsymbol{b}}(t)) + 2\alpha(A^\top\dot{\boldsymbol{u}}_\mathrm{N} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^\top\boldsymbol{u}_\mathrm{N} - \boldsymbol{b}(t)) = \boldsymbol{0}$$

$$\Downarrow$$

$$-A^\top\dot{\boldsymbol{v}}_\mathrm{N} = -\ddot{\boldsymbol{b}}(t) + 2\alpha(A^\top\boldsymbol{v}_\mathrm{N} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^\top\boldsymbol{u}_\mathrm{N} - \boldsymbol{b}(t))$$

Lagrange equation

Canonical form

$$\dot{\boldsymbol{u}}_{\mathrm{N}} = \boldsymbol{v}_{\mathrm{N}}$$

$$\begin{bmatrix} M & -A \\ -A^{\top} & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_{\mathrm{N}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} \\ \boldsymbol{C}(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}) \end{bmatrix}$$

where

$$\boldsymbol{C}(\boldsymbol{u}_{\mathrm{N}}, \boldsymbol{v}_{\mathrm{N}}) = -\ddot{\boldsymbol{b}}(t) + 2\alpha(A^{\top}\boldsymbol{v}_{\mathrm{N}} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^{\top}\boldsymbol{u}_{\mathrm{N}} - \boldsymbol{b}(t))$$

any ODE solver can be applied to the canonical form

Implementation

two-dimentional finite element calculation on MATLAB

`http://www.ritsumei.ac.jp/~hirai/edu/common/soft_robotics/Physics_Soft_Bodies.html`

Classes : `NodalPoint`, `Triangle`, `Body`

Implementation

```
classdef NodalPoint
    properties
        Coordinates;
        Displacement;
        Velocity
    end
    methods
        function obj = NodalPoint(p)
            obj.Coordinates = p;
        end
    end
end
```

Implementation

```
classdef Triangle
    properties
        Vertices;
        Area;
        Thickness;
        Density; lambda; mu;
        vector_a; vector_b;
        u_x; u_y; v_x; v_y;
        Cauchy_strain;
        Green_strain;
        Partial_J_lambda; Partial_J_mu;
        Partial_Stiffness_Matrix;
        Partial_Inertia_Matrix;
        Partial_Gravitational_Vector;
    end
    methods
```

Implementation

```
classdef Body
    properties
        numNodalPoints; NodalPoints;
        numTriangles; Triangles;
        strain_potential_energy;
        gravitational_potential_energy;
        J_lambda; J_mu;
        Stiffness_Matrix;
        Inertia_Matrix;
        Gravitational_Vector;
    end
    methods
        function obj = Body(npoints, points, ntris, tris,
            obj.numNodalPoints = npoints;
            for k=1:npoints
                pt(k) = NodalPoint(points(:,k));
```

Implementation

methods of class `Triangle`

partial_derivaties calculating partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$

calculate_Cauchy_strain calculating Cauchy strain in the triangle

partial_strain_potential_energy strain potential energy stored in the triangle

calculate_Green_strain calculating Green strain in the triangle

partial_strain_potential_energy_Green_strain strain potential energy using Green strain

partial_gravitational_potential_energy gravitational potential energy stored in the triangle

partial_stiffness_matrix calculating partial stiffness matrix $K_{i,j,k}$

partial_inertia_matrix calculating partial inertia matrix $M_{i,j,k}$

partial_gravitational_vector calculating partial gravitational vector $\boldsymbol{g}_{i,j,k}$

Implementation

methods of class Body

total_strain_potential_energy calculating strain potential energy stored in the body

total_strain_potential_energy_Green_strain strain potential energy using Green strain

total_gravitational_potential_energy gravitational potential energy stored in the body

calculate_stiffness_matrix calculating stiffness matrix K

calculate_inertia_matrix calculating inertia matrix M

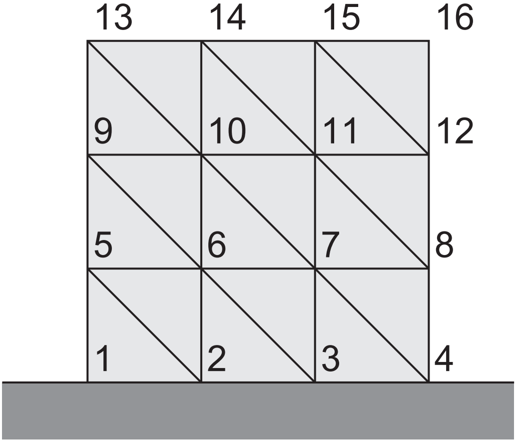
calculate_gravitational_vector calculating gravitational vector $\boldsymbol{g}$

constraint_matrix constraint matrix when specified nodal points are fixed

draw draw the shape of the body

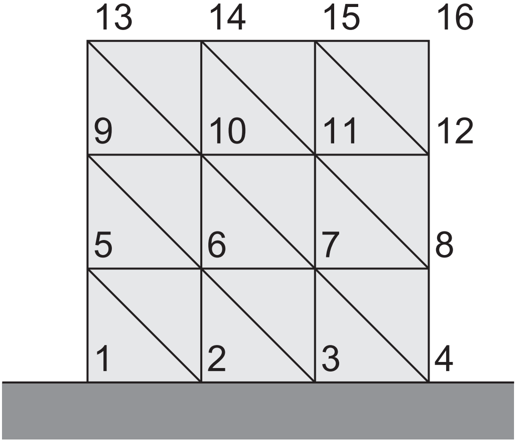
Example (dynamic simulation)

two-dimensional square soft body of width w
Young's modulus E, viscous modulus c, density ρ
divide square into $3 \times 3 \times 2$ triangles



Example (dynamic simulation)

$[\,0,\ t_{push}\,]$	fix the bottom & push $\mathrm{P}_{14}\mathrm{P}_{15}$ downward
$[\,t_{push},\ t_{hold}\,]$	fix the bottom & keep $\mathrm{P}_{14}\mathrm{P}_{15}$
$[\,t_{hold},\ t_{end}\,]$	fix the bottom & release $\mathrm{P}_{14}\mathrm{P}_{15}$



Example (dynamic simulation)

$[\,0,\ t_{push}\,]$ pushing velocity v_{push}

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \boldsymbol{0}$$

$$\boldsymbol{u}_{14} = \boldsymbol{u}_{15} = \boldsymbol{0} + \boldsymbol{v}_{push} t$$

where $\boldsymbol{v}_{push} = [\,0,\ -v_{push}\,]^{\top}$

$$A^{\top} = \begin{bmatrix} I & & & & \cdots & & \\ & I & & & \cdots & & \\ & & I & & \cdots & & \\ & & & I & \cdots & & \\ & & & & \cdots & I & \\ & & & & \cdots & & I \end{bmatrix}$$

1 2 3 4 14 15-th block columns

Example (dynamic simulation)

$[\,0,\ t_{push}\,]$

note

$$A^{\top}\boldsymbol{u}_{\rm N} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_{14} \\ \boldsymbol{u}_{15} \end{bmatrix}$$

speficies nodal points under constraints

Example (dynamic simulation)

$[\,0,\ t_{push}\,]$

$$\boldsymbol{b}(t) = \boldsymbol{b}_0 + \boldsymbol{b}_1 t$$

where

$$\boldsymbol{b}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{v}_{push} \\ \boldsymbol{v}_{push} \end{bmatrix}$$

note $\dot{\boldsymbol{b}}(t) = \boldsymbol{b}_1$ and $\ddot{\boldsymbol{b}}(t) = \mathbf{0}$, yielding

$$\boldsymbol{C}(\boldsymbol{u}_{\rm N}, \boldsymbol{v}_{\rm N}) = 2\alpha(A^\top \boldsymbol{v}_{\rm N} - \boldsymbol{b}_1) + \alpha^2(A^\top \boldsymbol{u}_{\rm N} - (\boldsymbol{b}_0 + \boldsymbol{b}_1 t))$$

Example (dynamic simulation)

$[\, t_{push},\ t_{hold}\,]$

$$
\boldsymbol{b}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ v_{push} t_{push} \\ v_{push} t_{push} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}
$$

Example (dynamic simulation)

$[\, t_{hold},\ t_{end} \,]$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \mathbf{0}$$

$$A^\top = \begin{bmatrix} I & & & & \cdots \\ & I & & & \cdots \\ & & I & & \cdots \\ & & & I & \cdots \end{bmatrix}$$

$$\boldsymbol{b}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

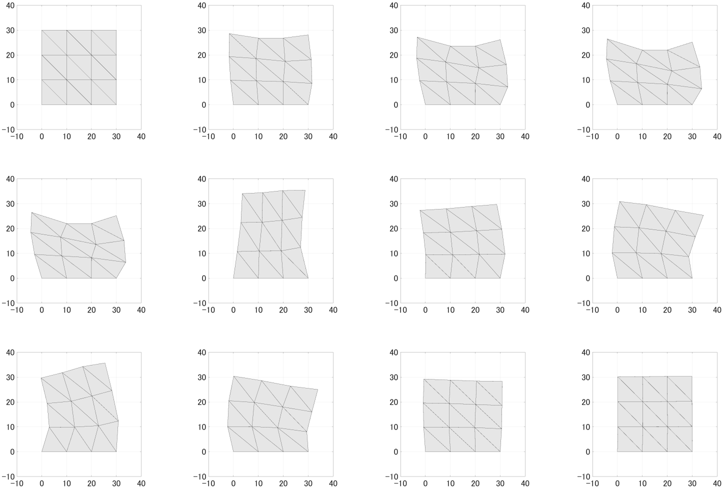
Example (dynamic simulation)

```
% Dynamic deformation of an elastic square object (4&time
% g, cm, msec

addpath('../two_dim_fea');
addpath('../two_dim_static');

width = 30; height = 30; thickness = 1;
m = 4; n = 4;
[points, triangles] = rectangular_object(m, n, width, hei

Young = 10.0; c = 0.04*Young; nu = 0.48; density = 1.00;
Epfloor = 0.02; cpfloor = 0;
[lambda, mu] = Lame_constants(Young, nu);
[lambdav, muv] = Lame_constants(c, nu);

npoints = size(points,2);
```

Example (dynamic simulation)

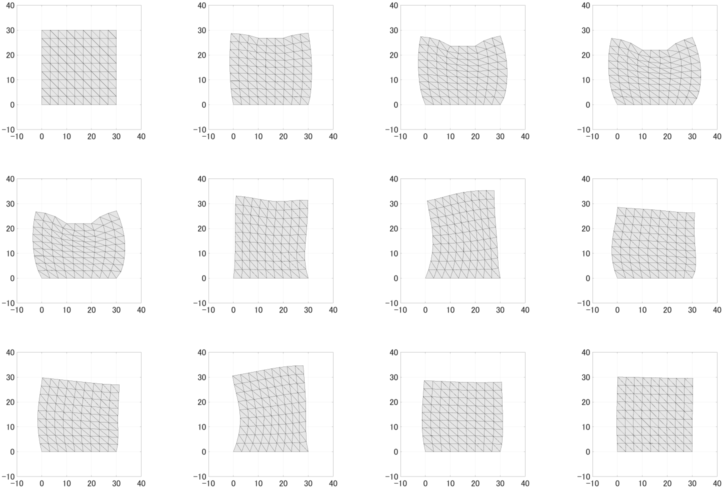
```
% holding top region
b0 = [ zeros(2*4,1); 0; -vpush*tp; 0; -vpush*tp ];
b1 = zeros(2*6,1);
interval = [tp, tp+th];
qinit = q_push(end,:);
square_object_hold = @(t,q) square_object_constraint_para
[time_hold, q_hold] = ode23tb(square_object_hold, interva

% releasing top region
A = elastic.constraint_matrix([1,2,3,4]);
b0 = zeros(2*4,1);
b1 = zeros(2*4,1);
interval = [tp+th, tp+th+tf];
qinit = q_hold(end,:);
square_object_free = @(t,q) square_object_constraint_para
[time_free, q_free] = ode23tb(square_object_free, interva
```

Example (dynamic simulation)



simulation movie

Example (dynamic simulation)



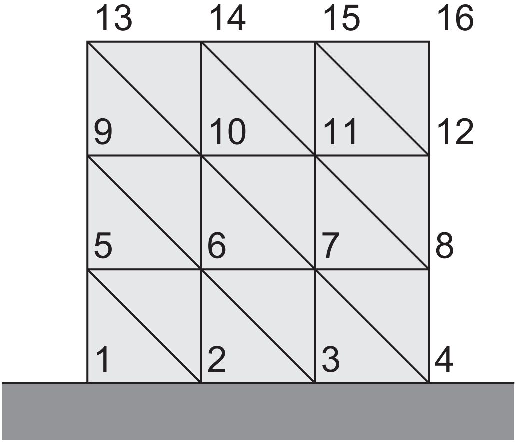
simulation movie

Example (dynamic simulation)

two-dimensional square soft body of width w

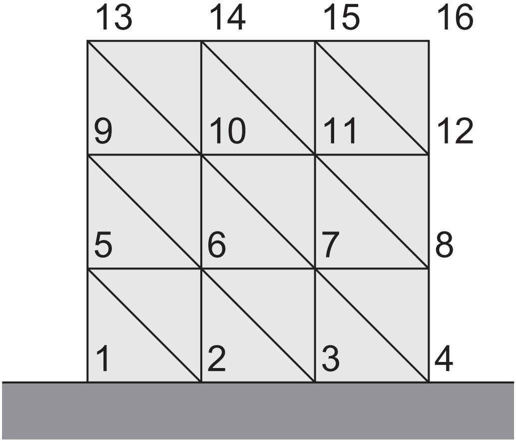
Young's modulus E, viscous modulus c, density ρ

divide square into $3 \times 3 \times 2$ triangles



Example (dynamic simulation)

$[\,0,\ t_{push}\,]$	fix the bottom & push $\mathrm{P}_{14}\mathrm{P}_{15}$ downward
$[\,t_{push},\ t_{hold}\,]$	fix the bottom & keep $\mathrm{P}_{14}\mathrm{P}_{15}$
$[\,t_{hold},\ t_{end}\,]$	free (reaction force by penalty method)



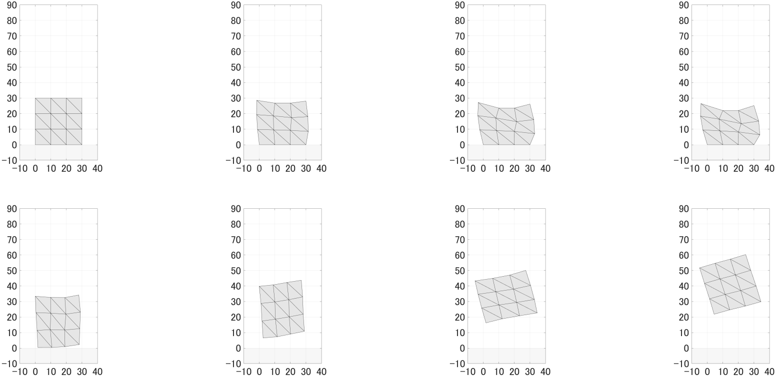
Example (dynamic simulation)

```
% Jumping of an elastic square object (4&times;4)
% g, cm, msec

addpath('../two_dim_fea');
addpath('../two_dim_static');

width = 30; height = 30; thickness = 1;
m = 4; n = 4;
[points, triangles] = rectangular_object(m, n, width, hei

Young = 10.0; c = 0.04*Young; nu = 0.48; density = 1.00;
Epfloor = 0.02; cpfloor = 0;
[lambda, mu] = Lame_constants(Young, nu);
[lambdav, muv] = Lame_constants(c, nu);

npoints = size(points,2);
```

Example (dynamic simulation)

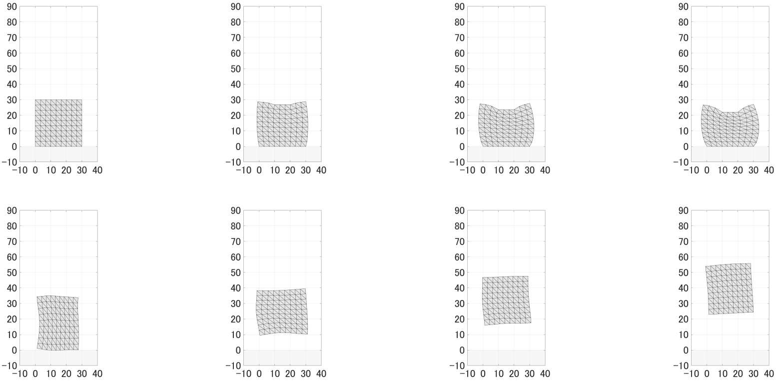
```
% holding top region
b0 = [ zeros(2*4,1); 0; -vpush*tp; 0; -vpush*tp ];
b1 = zeros(2*6,1);
interval = [tp, tp+th];
qinit = q_push(end,:);
square_object_hold = @(t,q) square_object_constraint_para
[time_hold, q_hold] = ode23tb(square_object_hold, interva

% releasing all constraints
floor_force = @(t,npoints,un,vn) floor_force_param(t,npoi
interval = [tp+th, tp+th+tf];
qinit = q_hold(end,:);
square_object_free = @(t,q) square_object_free_param(t,q,
[time_free, q_free] = ode23tb(square_object_free, interva

time = [time_push; time_hold; time_free];
```

Example (dynamic simulation)



jump simulation movie

Example (dynamic simulation)



jump simulation movie

Example (dynamic simulation)

- motion and deformation can be simulated properly
- results depend on mesh and include artifacts
- finer mesh yields better result but needs more computation time

Handouts

Sample programs (MATLAB) are available at:

`http://www.ritsumei.ac.jp/~hirai/edu/common/soft_robotics/Physics_Soft_Bodies.html`

Report due date: 23:59, July 23 (Tues)

Simulate the deformation of an elastic body

$\mathrm{P}_2\mathrm{P}_3$ is fixed to the floor.

P_1 and P_4 may slide on the floor.

$[\,0,\ t_{push}\,]$ push $\mathrm{P}_{14}\mathrm{P}_{15}$ downward

$[\,t_{push},\ t_{hold}\,]$ keep $\mathrm{P}_{14}\mathrm{P}_{15}$

$[\,t_{hold},\ t_{end}\,]$ release $\mathrm{P}_{14}\mathrm{P}_{15}$