On rigidity of Lie foliations

Gaël Meigniez* Hiraku Nozawa**

*Laboratoire de Mathématiques de Bretagne Atlantique
Université de Bretagne-Sud

**Department of Mathematical Sciences
College of Science & Engineering
Ritsumeikan University

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G : a connected Lie group

M : a smooth manifold

Definition

A G-Lie foliation of M is a foliation of M endowed with a transverse (G, G)-structure.

M : a closed manifold

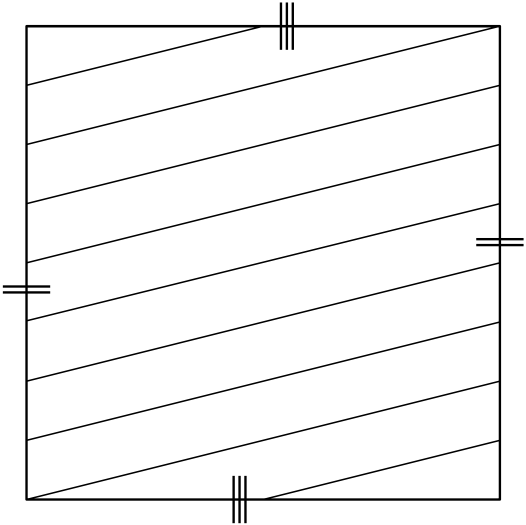
Theorem (Fedida)

Any G-Lie foliation $\mathcal{F}$ of M is given by

- *a homom* $\mathrm{hol} : \pi_1 M \longrightarrow G$ *and*
- *a fiber bundle* $\mathrm{dev} : \widetilde{M}^{\mathrm{univ}} \longrightarrow G$*, which is $\pi_1 M$-equivariant.*

$$\mathrm{dev}(c \cdot x) = \mathrm{hol}(c) \cdot_G \mathrm{dev}(x) \quad (\forall c \in \pi_1 M, \ \forall x \in \widetilde{M}^{\mathrm{univ}})$$

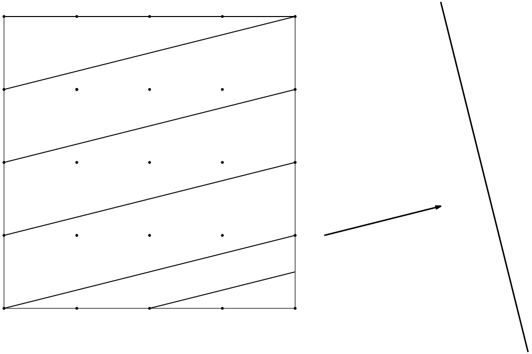
Example 1: Linear flows on T^2



$$\alpha := \text{slope}$$

Example 1: Linear flows on T^2

Let dev : $\mathbb{R}^2 \longrightarrow \mathbb{R}$ be the projection along the lines of slope α s.t. $\mathrm{dev}(0,0) = 0$.
Let $\mathrm{hol} = \mathrm{dev}\,|_{\mathbb{Z}^2} : \pi_1 T^2 \cong \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$.



H, G : Lie groups,

$K < H$: a Lie subgroup,

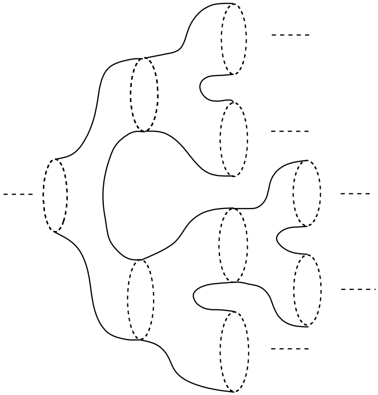
$\Gamma < H \times G$: a lattice,

$M = K\backslash H \times G/\Gamma$ has a G-Lie fol given by

- hol : $\pi_1 M \cong \Gamma \hookrightarrow H \times G \to G$ and
- dev $=$ proj$_2$: $K\backslash H \times G \longrightarrow G$.

Example 3: Hector-Matsumoto-Meigniez's example

$\exists$ an $\mathrm{SL}(2;\mathbb{R})$-Lie fol on a closed 5-mfd whose leaves are Cantor's tree surfaces.



A list of authors on the classification of G-Lie foliations in the case where G is solvable:

- Haefliger + Ghys,
- Caron-Carrière,
- Matsumoto-Tsuchiya,
- Meigniez.

In the semisimple case, only one result due to Zimmer has been known .

X : a symmetric sp. of non-cpt type (i.e., $X = \prod_i H_i/K_i$) s.t. $\mathrm{rank}_{\mathbb{R}} H_i > 1$.
$(M, \mathcal{F})$: a closed mfd with a minimal G-Lie fol.

Theorem (Zimmer)

If M admits a Riemannian metric s.t. $\forall L \in \mathcal{F}$ is isometric to X, then

- *$\exists$ a homogeneous G-Lie fol $(M_0, \mathcal{F}_0)$ and*
- *$\exists$ a C^∞-map $h : M \to M_0$ s.t. $\mathcal{F} = h^*\mathcal{F}_0$.*

Moreover, G is semisimple and $\mathrm{hol}(\pi_1 M)$ is arithmetic.

Main theorem

X : a symmetric sp. of non-cpt type (i.e., $X = \prod_i H_i/K_i$) s.t. $H_i \neq SL(2;\mathbb{R})$.
$(M, \mathcal{F})$: a closed mfd with a minimal G-Lie fol.

Theorem (Meigniez-N.)

If M admits a Riemannian metric s.t. $\forall L \in \mathcal{F}$ is isometric to X, then $(M, \mathcal{F})$ is homogeneous, i.e.,

- *$\exists$ a homogeneous G-Lie fol. $(M_0, \mathcal{F}_0)$ and*
- *$\exists$ a homeo $h : M \to M_0$ s.t. $\mathcal{F} = h^*\mathcal{F}_0$.*

Moreover, G is semisimple.

Combining with Margulis' theorems, we get the following corollaries.

Corollary

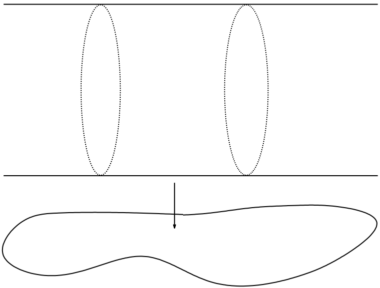
$\pi_1 M$ is isomorphic to a cocpt lattice in $G \times \prod H_i$, which is superrigid and arithmetic.

Corollary

$(M, \mathcal{F})$ is locally rigid in the sense of deformation theory.

Outline of the proof of the main theorem

Slogan: "Rigidity comes from the boundary of infinity."



Consider the case of $X = \mathbb{H}^n$. The key step is to observe that $\pi_1 M$ acts on the leafwise boundaries conformally, which yields

$$\varphi : \pi_1 M \longrightarrow \mathrm{Conf}(\partial \mathbb{H}^n) = H.$$

By using $\varphi \times \mathrm{hol} : \pi_1 M \longrightarrow H \times G$, construct a homogeneous G-Lie fol $(M_0, \mathcal{F}_0)$. Obtain a homeo $h : M \longrightarrow M_0$ s.t. $\mathcal{F} = h^* \mathcal{F}_0$ by using the classifying map $M \longrightarrow M_0$ of $\mathcal{F}$.

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Thank you for your attention!!