Parameter estimation for neuron models

Isao Tokuda*, Ulrich Parlitz†, Lucas Illing**, Matthew Kennel** and Henry Abarbanel‡

*Department of Computer Science and Systems Engineering, Muroran Institute of Technology, Muroran, Hokkaido 050-8585, Japan
†Drittes Physikalisches Institut, Universität Göttingen, Bürgerstraβe 42-44, D-37073 Göttingen, Germany
** Institute for Nonlinear Science, University of California, San Diego, La Jolla, CA 92093-0402
‡ Department of Physics and Marine Physical Laboratory, University of California, San Diego, La Jolla, CA 92093-0402

Abstract. Methods for estimating parameters of the Hindmarsh-Rose (HR) neuron model from a single time series are investigated. Two approaches, (1) synchronization based parameter estimation and (2) adaptive observer, are presented. Both methods are applied to membrane potential data recorded from a single lateral pyloric neuron synaptically isolated from other neurons.

INTRODUCTION

In neuroscience research, nonlinear dynamical systems approach to understanding neuronal activity has recently drawn a great deal of attention [1]. The methods of nonlinear time series analysis [2, 3] applied to isolate neurons from the stomatogastric ganglion of the California spiny lobster *Panuliru interruptus* revealed that the number of degrees of freedom in their membrane potential oscillation ranges typically from three to five [4]. Based on this observation, models of the action potential activity in this biological system have been developed using the framework of Hindmarsh and Rose (HR) [5]. The HR model is represented by three or four dimensional ordinary differential equations (ODEs), in which the complicated current-voltage relationships of the conductance based models [6] are replaced by polynomials in the dynamical variables. The HR model has been further implemented into an analog electronic neuron (EN) [7], whose properties are designed to emulate the membrane voltage characteristics of individual neurons. Using the EN, synchronization and regularization properties between EN and real biological neuron have been reported [8].

Although the numerical and the analog circuit studies have already demonstrated feasibility and consistency of the HR model with the real neurons, the model equations yet have some free parameters. For a systematic construction of the HR model that best matches individual neurons, parameter estimation of the model equations from real data is desired. Here, we present two approaches, (1) synchronization based parameter estimation and (2) adaptive observer, to matching dynamical variable of the HR model to membrane potential data recorded from real neurons.

SYNCHRONIZATION METHOD

Suppose we have a single time series $s(t)$ measured from an unknown dynamical system. Our goal is to find parameters of a model dynamical system $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y},\mathbf{a})$ ($\mathbf{F}: R^d \times R^m \to R^d$) so that the dynamics of $\mathbf{y}$ is commensurate with the unknown dynamics that revealed s. The assumption here is that the general form of $\mathbf{F}$ has been derived as a physiological neuron model but there remain parameter values that are matched to the data s recorded from real neurons. Our approach [9, 10] is to minimize, over free parameters $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{y}_0$,

$$J = \sum_{i=1}^{N} \frac{1}{2}[y_1(t_i) - \Phi_1(s(t_i),\mathbf{b})]^2 \tag{1}$$

subject to the constraints

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{F}(\mathbf{y},\mathbf{a}) + \mathbf{K}\cdot[\Phi(s,\mathbf{b}) - \mathbf{y}], \quad \mathbf{y}(t_1) = \mathbf{y}_0 && (2)\\ \mathbf{K} &= \begin{pmatrix} K & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} && (3) \end{aligned}$$

where $\Phi(s,\mathbf{b})$ is a smooth potentially nonlinear static map with free parameters $\mathbf{b}$.

There is a significant literature in numerical least-squares parameter estimation in ODEs with unperturbed constraint equation, *i.e* $K = 0$. With long chaotic orbit, this problem becomes progressively ill-conditioned. On account of the exponential separation of trajectories the error surface J grows increasingly bumpy with very large gradients $\partial J/\partial(\mathbf{a},\mathbf{b},\mathbf{y}_0)$. Our idea is that, by inducing dynamical synchronization between the signal and the model ($y_1 \approx \Phi_1(s,\mathbf{b})$), the error surface will be smoothed and regularized. With this constrained dynamics, trajectory of the model equations does not run away from the signal even with chaos and numerical process of minimizing the error becomes stable. Among various techniques to minimize the constrained error function J, the quasi-Newton method [11] combined with line search is employed. For computation of the gradients $\partial J/\partial(\mathbf{a},\mathbf{b},\mathbf{y}_0)$, variational equations of the constrained dynamics (2) are solved numerically. In order that the model finally reproduces qualitatively similar dynamics as the forcing signal in an autonomous condition, the forcing term of equation (2) must be weakened as $J \to 0$. We realize this by eventually decreasing the coupling strength K nearly to zero in the optimization process.

Let us apply the synchronization technique to 4-dimensional version of the HR equations [7, 8]

$$\begin{aligned} \frac{1}{\tau}\cdot\frac{dy_1}{dt} &= y_2 + b\cdot y_1^2 - y_1^3 - d\cdot y_3 + I + K\cdot s && (4)\\ \frac{1}{\tau}\cdot\frac{dy_2}{dt} &= -y_2 - f\cdot x_1^2 - g\cdot y_4 && (5)\\ \frac{1}{\tau}\cdot\frac{dy_3}{dt} &= \mu\cdot[-y_3 + y_1] && (6)\\ \frac{1}{\tau}\cdot\frac{dy_4}{dt} &= \nu\cdot[-y_4 + y_2]. && (7) \end{aligned}$$

This HR model has 8 free parameters $\{b,d,I,f,g,\mu,\nu,\tau\}$ (including time scaling parameter τ to match with the real time), where all the 8 parameters were optimized by the synchronization method. For simplicity, the transformation was set to be an identity map, *i.e.* $\Phi_1(s) = s$. As the forcing signal s, membrane potential data $\{s(i\cdot\Delta t) : i = 1,\ldots,22000\}$ (sampling frequency: $\Delta t = 0.5$ msec) recorded from a single lateral pyloric neuron synaptically isolated from other neurons were used. As initial condition of the free parameters, parameter values of ref. [8] were used. The coupling strength was initially set as $K = 1$ and with every 250 optimization steps weakened as $K = 0.9$, $K = 0.8$, $\ldots$, $K = 0.1$, $K = 0.1\times0.5$, $\ldots$, $K = 0.1\times0.5^9$.

Figure 1 shows simultaneous drawing of the real neuron data and the HR dynamics without the forcing term, *i.e.* $K = 0$. Qualitatively, similar behavior has been reproduced by the HR model. The HR dynamics obtained by the synchronization method is chaotic in the sense that it has a positive first Lyapunov exponent $\lambda_1 = 0.0027$ with a Lyapunov dimension $d_{KY} = 2.001$. This is in a good agreement with the real data, which was shown to have a positive Lyapunov exponent by nonlinear time series analysis [4]. Quantitative disagreement between the HR model and the real data such as increasing and then decreasing spike amplitudes present in the data but not in the model might be due to (a) limitation of the simplified model to describe exact physiological process and (b) recording or dynamical noise.

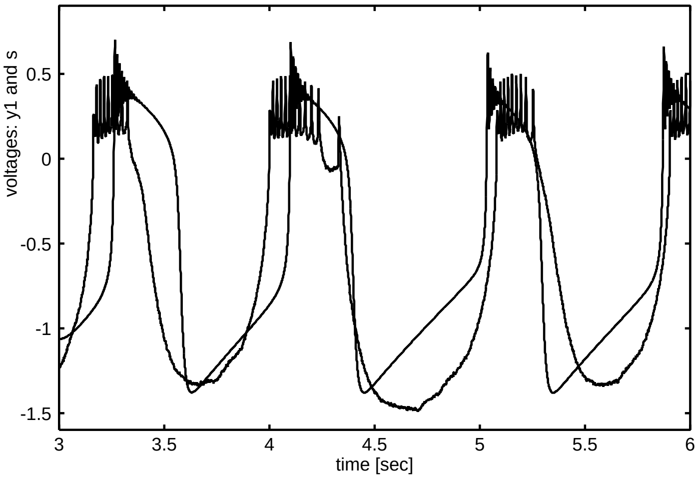

FIGURE 1. Simultaneous drawing of real neuron data (thick line) and HR dynamics (thin line).

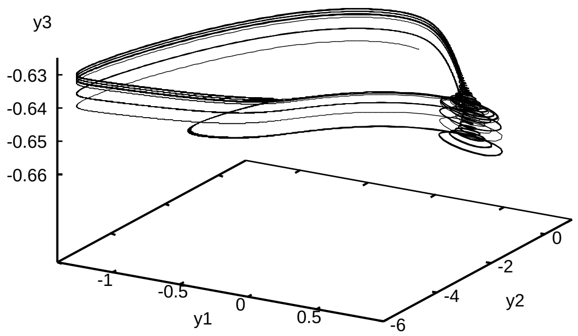

FIGURE 2. Chaotic attractor in (y_1, y_2, y_3)-space obtained by the synchronization method.

ADAPTIVE OBSERVER

In this section we present an adaptive observer, *i.e.* a dynamical system that is driven by the given time series and that possesses additional ODEs governing (slow) variations of model parameters. Once the parameters have converged to the "right" values the model oscillations synchronize with the driving time series.

The observer is based on a 3-dimensional HR model

$$\begin{aligned} \dot{v} &= F(v) + y - z \\ \dot{y} &= b_1 v + b_2 v^2 - b_3 y \\ \dot{z} &= b_4 v - b_5 z \end{aligned} \tag{8}$$

where $F(v) = a_0 + a_1 v + a_2 v^2 + a_3 v^3$. In the above normalized model, redundant parameters present in the original 3-dimensional HR equations [5] have been removed. With this parametrization the model is able to adapt to arbitrary time scale and range of the voltage variable v. To build an observer some extra terms are added to implement driving by an external voltage s

$$\dot{v} = F(s) + y - z + k(s - v) \tag{9}$$

$$\dot{y} = b_1 s + b_2 s^2 - b_3 y + s - v \tag{10}$$

$$\dot{z} = b_4 s - b_5 z - (s - v) \tag{11}$$

and suitable ODEs for all parameters $a_0, \ldots, a_3$ and $b_1, \ldots, b_5$.

$$\dot{a}_0 = s - v \tag{12}$$

$$\dot{a}_1 = (s - v)s \tag{13}$$

$$\dot{a}_2 = (s - v)s^2 \tag{14}$$

$$\dot{a}_3 = (s - v)s^3 \tag{15}$$

$$\dot{b}_1 = -0.1(s - v) \tag{16}$$

$$\dot{b}_2 = 0.1(s - v)vz \tag{17}$$

$$\dot{b}_3 = -0.1(s - v)y \tag{18}$$

$$\dot{b}_4 = -0.1(s - v)z \tag{19}$$

$$\dot{b}_5 = -0.1(s - v)yz. \tag{20}$$

Note that the right hand sides of all parameter ODEs vanish in the case of perfect synchronization $v = s$. Figure 3 shows the temporal evolution of the measured signal s and the corresponding voltage variable of the HR-observer (9)-(20). The underlying parameter estimation process is shown in Fig. 4. As can be seen in Fig. 3 model variable v converges to the driving neuron voltage s with a small synchronization error $s - v$ (Fig. 3a and section Fig. 3b). The fact that some of the estimated parameters do not converge to fixed values but oscillate with small amplitude may be interpreted as an indication for some limited capability of the HR-model to describe the dynamics of the real neuron quantitatively.

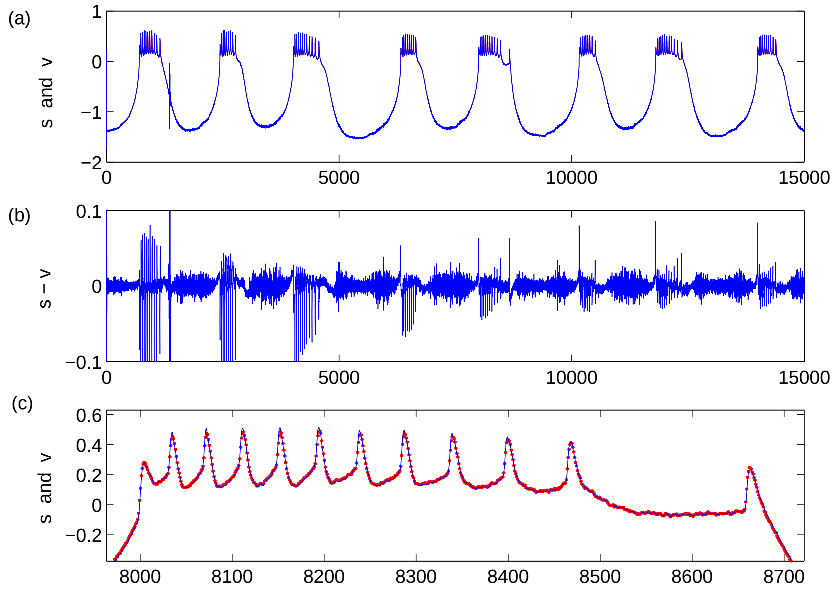

FIGURE 3. (a) Synchronization of the measured voltages s from a real neuron and the corresponding variable v of the Hindmarsh Rose observer (9)-(20). Within the graphical resolution both curves are not distinguishable. (b) Synchronization error $s - v$ vs. sample number. (c) Section of (a) where the driving signal s is plotted using filled circles.

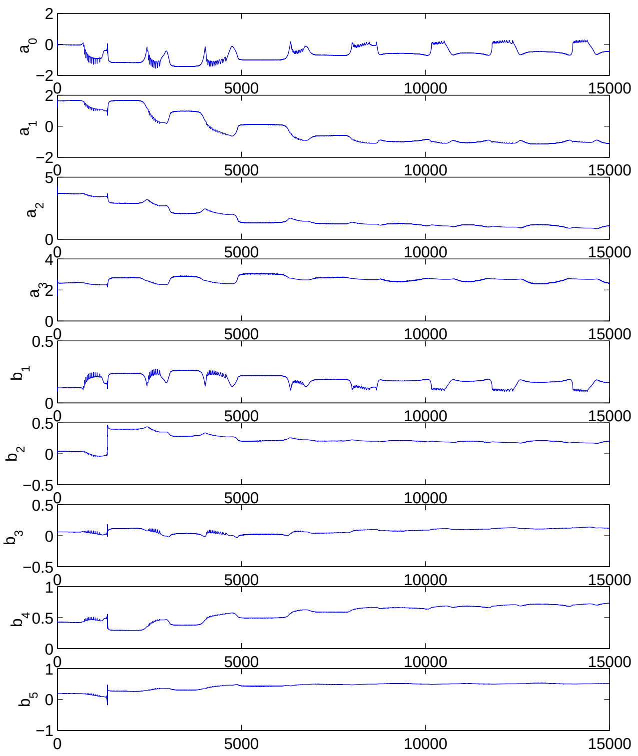

FIGURE 4. Model parameters $a_0, \ldots, a_3$ and $b_1, \ldots, b_5$ vs. sample number (time). After some transient the parameters converge to fixed values or oscillate with small amplitude.

CONCLUSIONS

Two approaches have been presented for parameter estimation of the HR models to match with the real neuron data. Both the synchronization technique and the adaptive observer technique utilize synchronized dynamics between the model and the data so as to suppress the modeling error expansion induced by chaotic property inherent in the neuron data. Disadvantages of the synchronization technique are (i) it has a limited basin of attraction that leads to feasible solutions especially when the number of the free parameters is large and (ii) its convergence is rather slow. The adaptive observer, on the other hand, has fast convergence property. There is, however, no systematic way to construct an adaptive observer to optimize all model parameters. Except for some special cases, global convergence of the adaptive observer is not guaranteed.

Alternative approach to the parameter estimation is the multiple-shooting (MS) method [12], which is one of the standard methods for parameter estimation of ODEs. According to our numerical study, as the number of the free parameters is increased, basin of attraction leading to feasible solutions gets much smaller in the MS method and further modifications are necessary in order to apply for the real data. In our future work, combination of the MS method with the idea of synchronization will be examined. Considering the effect of dynamical and observational noise, extended Kalman filtering technique for stochastic nonlinear modeling [13] will be also investigated.

Limitation of the HR modeling of the real neuron data might also be due to an incapability of the simplified HR model to reproduce complicated physiological process. Modified models with (a) nonlinear transformation for adjusting spiking amplitudes to bursting amplitudes and (b) addition of delayed feedback process will be considered in our next study.

REFERENCES

1. H. D. I. Abarbanel and M. I. Rabinovich, Current Opinion in Neurobio. **11**, 423 (2001).
2. H. D. I. Abarbanel, *Analysis of Observed Chaotic Data*, (Springer, New York, 1996).
3. H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis*, (Cambridge University Press, Cambridge, 1997).
4. H. D. I. Abarbanel *et al.*, Neural Comput. **8**, 1567 (1996).
5. J.L. Hindmarsh and R.M. Rose, Proc. R. Soc. Lond. B **221**, 87 (1984).
6. A. L. Hodgkin and A. F. Huxley, J. Physiol. (London) **117**, 500 (1952).
7. A. I. Selverston *et al.*, J. Physiol. (London) **94**, 357 (2000).
8. R. D. Pinto *et al.*, Phys. Rev. E **62**, 2644 (2000).
9. U. Parlitz, L. Junge, and L. Kocarev, Phys. Rev. E **54**, 6253 (1996).
10. H. D. I. Abarbanel *et al.*, Presentation Material from Dynamic Days (2001).
11. D. G. Luenberger, *Linear and Nonlinear Programming* (Addison-Wesley, 1973).
12. E. Baake, M. Baake, H. G. Bock, and K. M. Briggs, Phys. Rev. A **45**, 5524 (1992).
13. D. M. Walker, Phys. Lett. A **249**, 209 (1998).